

Topological and algebraic structures on the ring of Fermat reals

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Abstract

The ring of Fermat reals is an extension of the real field containing nilpotent infinitesimals, and represents an alternative to Synthetic Differential Geometry in classical logic. In the present paper, our first aim is to study this ring from using standard topological and algebraic structures. We present the Fermat topology, generated by a complete pseudo-metric, and the omega topology, generated by a complete metric. The first one is closely related to the differentiation of (non standard) smooth functions defined on open sets of Fermat reals. The second one is connected to the differentiation of smooth functions defined on infinitesimal sets. Subsequently, we prove that every (proper) ideal is a set of infinitesimals whose order is less than or equal to some real number. Finally, we define and study roots of infinitesimals. A computer implementation as well as an application to infinitesimal Taylor formulas with fractional derivatives are presented.

Keywords: Fermat reals, nilpotent infinitesimals, ideals, roots

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1 Introduction

If mathematics is the language of nature, one can imagine that the more results are discoverable and describable using a given part of mathematics, the more faithfully that language will correspond to some given part of nature. We can hence imagine that there is a sort of weak isomorphism between that language and the corresponding part of nature. Therefore, if two different languages are able to describe faithfully the same part of nature, we can also think that these two languages are, in some way, isomorphic to each other. So, because we are able to use mathematical analysis and actual infinitesimals as

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languages to describe nature, we can imagine that using the first one it would be possible to obtain a rigorous and modern model of the informal use of infinitesimals. If this idea is in some way correct, this should actually be feasible without any non trivial background of mathematical logic. The theory of Fermat reals represents a possible formalization of this philosophical idea. Other possible approaches following this line of thought are: Weil functors [17, 18], Levi-Civita fields [20, 25, 26], Surreal numbers [7, 8], geometries over a general base ring [2], or the ring of Colombeau generalized numbers [5, 6, 22]. Classical approaches requiring a non trivial background of mathematical logic are Non-standard Analysis [24] and Synthetic Differential Geometry [15, 19, 21]. In case the above philosophical idea sounds natural to the reader, an open problem is to understand, from a mathematical, social or historical point of view, why the latter theories, i.e. those requiring a non trivial background of mathematical logic, seem more powerful than the former ones.

The ring ${}^\bullet\mathbb{R}$ of Fermat reals can be defined and studied using elementary calculus only ([13]). It extends the field \mathbb{R} of real numbers and contains nilpotent infinitesimals, i.e. $h \in {}^\bullet\mathbb{R}_{\neq 0}$ such that $h^n = 0$ for some $n \in \mathbb{N}_{>1}$. The methodological thread followed in the development of the theory of Fermat reals has always been guided by the necessity to obtain a good dialectic between formal properties and their informal interpretations. Indeed, the ring is totally ordered and geometrically representable ([11, 12, 9]), to cite some examples.

Every Fermat real $x \in {}^\bullet\mathbb{R}$ can be written, in a unique way, as

$$x = {}^\circ x + \sum_{i=1}^N \alpha_i \cdot dt_{a_i}, \quad (1.1)$$

where ${}^\circ x, \alpha_i, a_i \in \mathbb{R}$ are standard reals, $a_1 > a_2 > \dots > a_N \geq 1$, $\alpha_i \neq 0$, and where dt_a verifies the following properties

$$\begin{aligned} dt_a \cdot dt_b &= dt_{\frac{ab}{a+b}} \\ (dt_a)^p &= dt_{\frac{a}{p}} \quad \forall p \in \mathbb{R}_{\geq 1} \\ dt_a &= 0 \quad \forall a \in \mathbb{R}_{<1}. \end{aligned} \quad (1.2)$$

The expression (1.1) is called the *decomposition* of x , and the real number ${}^\circ x$ its *standard part*. The number $a_1 =: \omega(x)$ is called the *order* of x and represents the greatest infinitesimal appearing in its decomposition. In case $x \in \mathbb{R}$, i.e. $x = {}^\circ x$, we set $\omega(x) = 0$. We will also use the notations $\omega_i(x) := a_i$ and ${}^\circ x_i := \alpha_i$ for the i -th order and the i -th *standard part* of x ; $\omega_i(x) := 0$ if $x \in \mathbb{R}$. The order $\omega(-)$ has the following natural properties

$$\begin{aligned} \omega(x + y) &= \max[\omega(x), \omega(y)] \\ \frac{1}{\omega(x \cdot y)} &= \frac{1}{\omega(x)} + \frac{1}{\omega(y)}, \end{aligned}$$

whenever x, y are infinitesimals such that $x + y \neq 0$ or $x \cdot y \neq 0$, respectively.

Directly from (1.1) it is not hard to prove that if $k \in \mathbb{N}_{>1}$, then $x^k = 0$ iff $\omega(x) < k$. For $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, the ideal

$$D_a := \{x \in \bullet\mathbb{R} \mid {}^\circ x = 0, \omega(x) < a + 1\}$$

plays a fundamental role in k -th order Taylor formulas with nilpotent increments (so that the remainder is zero). Indeed, for $k \in \mathbb{N}_{\geq 1}$ we have that $D_k = \{x \in \bullet\mathbb{R} \mid x^{k+1} = 0\}$, and any ordinary smooth function $f : A \rightarrow \mathbb{R}$ defined on an open set A of \mathbb{R}^d can be extended to the set

$$\bullet A = \left\{x \in \bullet\mathbb{R}^d \mid {}^\circ x \in A\right\},$$

obviously obtaining a true extension, i.e. the same values at $x \in A$. The mentioned Taylor formula is therefore

$$\forall h \in D_k^d : f(x + h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \leq k}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x),$$

where $x \in A$ is a standard point, and $D_k^d = D_k \times \dots \times D_k$.

It may seem difficult to work in a ring with zero divisors, but the following properties permit to deal effectively with products of nilpotent infinitesimals (typically appearing in multidimensional Taylor formulas) and with cancellation laws:

$$h_1^{i_1} \cdot \dots \cdot h_n^{i_n} = 0 \iff \sum_{k=1}^n \frac{i_k}{\omega(h_k)} > 1$$

$$x \text{ is invertible} \iff {}^\circ x \neq 0$$

$$(\text{If } x \cdot r = x \cdot s \text{ in } \bullet\mathbb{R}, \text{ where } r, s \in \mathbb{R} \text{ and } x \neq 0) \implies r = s.$$

Finally, the ring $\bullet\mathbb{R}$ is totally ordered, and the order relation can be effectively decided, once again, starting from the decompositions: let $x, y \in \bullet\mathbb{R}$; if ${}^\circ x \neq {}^\circ y$, then

$$x < y \iff {}^\circ x < {}^\circ y. \tag{1.3}$$

Otherwise, if ${}^\circ x = {}^\circ y$, then

1. If $\omega(x) > \omega(y)$, then $x > y$ iff ${}^\circ x_1 > 0$, i.e. iff $x > 0$ (from (1.3))
2. If $\omega(x) = \omega(y)$, then

$$\begin{aligned} {}^\circ x_1 > {}^\circ y_1 &\implies x > y \\ {}^\circ x_1 < {}^\circ y_1 &\implies x < y. \end{aligned}$$

For example, $dt_3 - 3dt > dt > 0$, and $dt_a < dt_b$ if $a < b$. This quick summary of some algebraic and order properties of the ring $\bullet\mathbb{R}$ of Fermat reals can be considered as a first step toward its axiomatic description. A very simple model can already be guessed from the properties (1.1) and (1.2). Indeed, we first introduce the ring $\mathbb{R}_o[t]$ of little-oh polynomials, i.e. functions $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ that can be written as $x(t) = r + \sum_{i=1}^N a_i \cdot t^{\alpha_i} + o(t)$, as $t \rightarrow 0^+$, where $N \in \mathbb{N}$, $r, a_1, \dots, a_N \in \mathbb{R}$ and $\alpha_1, \dots, \alpha_N \in \mathbb{R}_{\geq 0}$. Then, in the ring $\mathbb{R}_o[t]$, we define the equivalence relation $x \sim y$ iff $x(t) = y(t) + o(t)$, for $t \rightarrow 0^+$, and $\bullet\mathbb{R} := \mathbb{R}_o[t] / \sim$ is the related quotient set.

For a more complete presentation and study of the model, see [13, 12, 9].

In [10], the Fermat-Reyes theorem, which is essential for the development of differential calculus on $\bullet\mathbb{R}$, is presented. The Fermat-Reyes theorem states the existence and uniqueness of the *smooth incremental ratio* of every smooth function $f : \bullet U \rightarrow \bullet\mathbb{R}$, that is existence and uniqueness of a function $r : \bullet\widetilde{U} \rightarrow \bullet\mathbb{R}$ satisfying

$$f(x+h) = f(x) + h \cdot r(x,h) \quad \text{in } \bullet\mathbb{R} \quad \forall (x,h) \in \bullet\widetilde{U},$$

where $\bullet\widetilde{U} := \{(x,h) \in \bullet\mathbb{R}^2 \mid \overrightarrow{[x, x+h]} \subseteq \bullet U\}$ is called the thickening of $\bullet U$ ([1, 10]). Here, the function f is more general than the extension from $U \subseteq \mathbb{R}$ to $\bullet U \subseteq \bullet\mathbb{R}$ of an ordinary smooth function defined on U and with values in \mathbb{R} . The function f is a *non standard smooth* (or, more simply, *smooth*) function, i.e., by definition, f can locally be written as

$$f(x) = \bullet\alpha(p, x) \quad \forall x \in \bullet V \cap \bullet U,$$

where $\alpha \in \mathcal{C}^\infty(W \times V, \mathbb{R}^t)$ is an ordinary smooth function defined on an open set of $\mathbb{R}^w \times \mathbb{R}^v$ and $p \in \bullet W$ is a w -dimensional parameter. The mentioned topology is that generated by extension of open sets, i.e. by sets of the form $\bullet U$. For example, $f(x) = dt + x$ is (non standard) smooth, but it is not the extension of an ordinary smooth function because $f(0) = dt \notin \mathbb{R}$, whereas any extension takes \mathbb{R} to itself. The Fermat-Reyes theorem is well framed in the cartesian closed category $\bullet\mathcal{C}^\infty$ of Fermat spaces, as this permits to develop a notion of smooth space and smooth function including also infinite dimensional spaces, e.g. function spaces like $\mathbf{Man}(M, N)$ or integral and differential operators. The definition of $\bullet\mathcal{C}^\infty$ is essentially a generalization of the notion of diffeological space ([14]), whose category \mathcal{C}^∞ is the domain of the Fermat functor $\bullet(-) : \mathcal{C}^\infty \rightarrow \bullet\mathcal{C}^\infty$. This functor generalizes the construction $\mathbb{R} \mapsto \bullet\mathbb{R}$ and displays very good preservation properties, closely related to intuitionistic logic. For more details see [10, 9].

The main aim of the present paper is to study the ring of Fermat reals using standard topological and algebraic structures. We will analyze interesting metric structures deeply related to the development of smooth calculus on $\bullet\mathbb{R}$. We will characterize the ideals of $\bullet\mathbb{R}$, confirming that they are of a simple (non-pathological) nature, due to the initial choice of the very well behaved family of little-oh polynomials. Finally we will show that, in spite of the presence of

infinitesimal numbers $h \in \bullet\mathbb{R}_{\neq 0}$ such that $h^2 = 0$, we can define powers h^p for every $p \in \mathbb{R}_{\geq 0}$ and hence, we have arbitrary roots with several good properties. This dialectic, between standard structures and the new ring $\bullet\mathbb{R}$, aims at presenting the theory of Fermat reals to a general mathematical audience, hoping that this will contribute to further studies of this interesting ring with infinitesimals.

2 Metric structures

The topology used to prove the above mentioned Fermat-Reyes theorem, the key theorem for the development of differential calculus of smooth functions defined on open sets, is that generated by extensions $\bullet U$ of open sets $U \subseteq \mathbb{R}^n$. In this approach, a subset $A \subseteq \bullet\mathbb{R}^n$ is open if it can be written as

$$A = \bigcup \{ \bullet U \subseteq A \mid U \text{ is open in } \mathbb{R}^n \}.$$

We will call the resulting topology the *Fermat topology*. In the present section, we want to show that in the ring $\bullet\mathbb{R}$ it is possible to define two interesting (pseudo) metric structures, corresponding to two different topologies. The first one is the Fermat topology, which can roughly be described as the best topology for sets having a “sufficient amount of standard points”, like, e.g., $\bullet U$. This connection between Fermat topology and standard points can be glimpsed by saying that the monad of a standard real $r \in \mathbb{R}$, i.e.

$$\mu(r) := \{x \in \bullet\mathbb{R} \mid {}^\circ x = r\} = \{r + h \mid h \in D_\infty\},$$

is the set of all the points which are limits of sequences with respect to the Fermat topology (which is not Hausdorff).

However, in sets of infinitesimals, like the ideal D_a , there is only one standard point, and indeed the best topology to study this kind of sets is not the Fermat one. Therefore, we will define a metric generating a finer topology, called the *omega topology*. When restricted to $\mu(r)$, the omega topology is naturally tied with the equality $=_k$ up to k -th order infinitesimals, i.e. $x =_k y$ iff ${}^\circ x = {}^\circ y$ and $\omega(x - y) \leq k$ (see [11, 9] for some properties and applications of this notion). It is worth noting that the equivalence relation $=_k$ is tied with differential calculus of smooth functions defined on infinitesimal sets like D_a . Indeed, in [9] it is proved that for a smooth function $f : D_a \rightarrow \bullet\mathbb{R}$ there always exist $m_1, \dots, m_n \in \bullet\mathbb{R}$, $n := [a]$ being the integer part of a , such that

$$f(h) = f(0) + \sum_{j=1}^n \frac{h^j}{j!} \cdot m_j \quad \forall h \in D_a.$$

Moreover, these m_j are unique up to k_j -th order infinitesimals

$$\left(\forall h \in D_a : \sum_{j=1}^n \frac{h^j}{j!} \cdot m_j = \sum_{j=1}^n \frac{h^j}{j!} \cdot \bar{m}_j \right) \implies m_j =_{k_j} \bar{m}_j \quad \forall j = 1, \dots, n,$$

where the k_j are defined by

$$\frac{1}{k_j} + \frac{j}{a+1} = 1.$$

This permits to define derivatives of smooth functions defined on infinitesimal sets. Therefore, it is worth noting that two standard metrics (i.e. with values in \mathbb{R} and not in ${}^\bullet\mathbb{R}$) are strictly related to the calculus of two different classes of smooth functions on the ring of Fermat reals.

To motivate the definition of our metrics on ${}^\bullet\mathbb{R}$, we can say that:

- We want to measure the distance between $x, y \in {}^\bullet\mathbb{R}$ on the basis of $\omega(x-y)$ and $|\circ x - \circ y|$ only.
- We want to extend the classical metric on the reals \mathbb{R} .

Definition 1. Let $x, y \in {}^\bullet\mathbb{R}$, then

1. $d_F(x, y) := |\circ x - \circ y|$
2. $d_\omega(x, y) := |\circ x - \circ y| + \omega(x - y)$.

Obviously, both d_F and d_ω extend the usual metric on \mathbb{R} ; moreover, if $x, y \in \mu(r)$, then $d_\omega(x, y) \leq k$ iff $x =_k y$. Finally, as we will see later, the idea to use also the i -th orders $\omega_i(x-y)$ to define other metrics on ${}^\bullet\mathbb{R}$ is not a successful one.

Theorem 2. *The Fermat and the omega metrics verify the following properties:*

1. $d_F : {}^\bullet\mathbb{R} \times {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$ is a pseudometric.
2. $d_\omega : {}^\bullet\mathbb{R} \times {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$ is a metric.
3. The d_ω -topology is finer than the d_F -topology.
4. The topology generated by d_F is the Fermat topology.
5. d_F and d_ω are not topologically equivalent.

Proof: The proof of 1 is direct. Concerning 2, we have that $d_\omega(x, y) = 0$ iff $\circ x = \circ y$ and $\omega(x - y) = 0$. The order $\omega(x - y)$ is zero iff $x - y \in \mathbb{R}$, i.e. $x - y =: r \in \mathbb{R}$. From $\circ x = \circ y$ it follows that $r = 0$ and hence the conclusion $x = y$. From the definition of order $\omega(-)$, the property $\omega(-z) = \omega(z)$ follows, and hence d_ω is symmetric. To prove the triangle inequality, we introduce the following lemma, which is a generalization of an analogous result already proved for $x, y \in D_\infty$ only (see e.g. [13]).

Lemma 3. *Let $x, y \in {}^\bullet\mathbb{R}$, then*

1. $x + y \notin \mathbb{R} \implies \omega(x + y) = \max[\omega(x), \omega(y)]$

$$2. \ x + y \in \mathbb{R} \implies \omega(x + y) = 0.$$

Therefore

$$\omega(x - y) \leq \omega(x - z) + \omega(z - y) \quad \forall z \in \bullet\mathbb{R}.$$

Using this lemma we have

$$\begin{aligned} d_\omega(x, y) &= |\circ x - \circ y| + \omega(x - y) \leq \\ &\leq |\circ x - \circ z| + |\circ z - \circ y| + \omega(x - z) + \omega(z - y) = \\ &= d_\omega(x, z) + d_\omega(z, y). \end{aligned}$$

Property 3 follows directly from the inequality $d_F(x, y) \leq d_\omega(x, y)$. To prove 4 let us firstly consider a d_F -open set \mathcal{U} . Then, for every $x \in \mathcal{U}$ we can find $r \in \mathbb{R}_{>0}$ such that

$$B_r(x; d_F) = \{y \in \bullet\mathbb{R} \mid |\circ x - \circ y| < r\} \subseteq \mathcal{U}. \quad (2.1)$$

Let $U := (\circ x - r, \circ x + r)_\mathbb{R} = \{s \in \mathbb{R} \mid |\circ x - s| < r\}$, then for every $y \in \bullet U$ we have $\circ y \in U$ and hence $y \in \mathcal{U}$ from (2.1). Therefore, $x \in \bullet U \subseteq \mathcal{U}$, that is \mathcal{U} is also open in the Fermat topology. Vice versa, let \mathcal{U} be an open set in the Fermat topology, then for every $x \in \mathcal{U}$ we can find an open set U of \mathbb{R} such that $x \in \bullet U \subseteq \mathcal{U}$. Therefore, $\circ x \in U$ and

$$(\circ x - r, \circ x + r)_\mathbb{R} \subseteq U \quad (2.2)$$

for some $r \in \mathbb{R}_{>0}$. So, for every $y \in B_r(x; d_F)$ we have $|\circ x - \circ y| < r$ and $\circ y \in U$ from (2.2). This implies $y \in \bullet U \subseteq \mathcal{U}$ and proves that $B_r(x; d_F) \subseteq \mathcal{U}$.

To prove 5 we can consider $\mathcal{U} = B_r(0; d_\omega)$, where $r > 1$. We want to show that every ball $B_s(0; d_F)$ is not contained in \mathcal{U} , that is

$$\exists x \in \bullet\mathbb{R} : |\circ x| < s, \quad |\circ x| + \omega(x) \geq r.$$

To prove this it suffices to show that $\omega(x) \geq r$ for some infinitesimal $x \in D_\infty$, which is trivially true: we can take, e.g., $x = dt_r$ whose order is $\omega(dt_r) = r$ because $r > 1$. \square

Proof of Lemma 3: Let $\delta x := x - \circ x$, $\delta y := y - \circ y$ be the infinitesimal parts of x, y , so that, directly from the definition of order, we have $\omega(x + y) = \omega(\delta x + \delta y)$. If $\delta x + \delta y = 0$, then $\omega(x + y) = \omega(0) = 0$. Otherwise, $\delta x + \delta y \neq 0$ and from Theorem 12 of [13] we have $\omega(\delta x + \delta y) = \max[\omega(\delta x), \omega(\delta y)] = \max[\omega(x), \omega(y)]$. Finally, it suffices to note that

$$\delta x + \delta y = 0 \iff x + y \in \mathbb{R}$$

to prove both properties 1 and 2.

To prove the stated inequality, we note that if $x + y \in \mathbb{R}$, then $\omega(x + y) = 0 \leq \omega(x) + \omega(y)$ because $\omega(-) \geq 0$. If $x + y \notin \mathbb{R}$, then either $\omega(x + y) = \omega(x) \leq \omega(x) + \omega(y)$ or $\omega(x + y) = \omega(y) \leq \omega(x) + \omega(y)$. From this, the conclusion follows, because $\omega(x + y) = \omega[(x - z) + (z - y)]$. \square

Definition 4. We will call ω -topology the topology generated by the metric d_ω . It can also be called the topology of the *order function* (to distinguish it from the topology of the order relation).

Is it possible to generalize using higher orders ω_i ?

It is very natural to try a generalization of the metric d_ω considering, e.g., also the information given by $\omega_2(x - y)$:

$$d_2(x, y) := |\circ x - \circ y| + \omega(x - y) + \omega_2(x - y).$$

However, an immediate problem is that higher orders $\omega_i(-)$, $i > 1$, do not share the good properties of $\omega(-)$. For example:

- If $x = dt$ and $y = dt_3 + dt_2$, then $\omega_2(x + y) = \omega_2(y)$, but if $y = dt_2$, then $\omega_2(x + y) = \omega(x)$.
- If $\omega(x) = \omega(y)$ and $\circ x_1 + \circ y_1 \neq 0$, then $\omega_2(x + y) = \max[\omega_2(x), \omega_2(y)]$, but if $\circ x_1 + \circ y_1 = 0$, $\omega_2(x) = \omega_2(y)$ and $\circ x_2 + \circ y_2 \neq 0$, then $\omega_2(x + y) = \max[\omega_3(x), \omega_3(y)]$.
- If $x = dt_6 + 2dt$, $y = dt_7 + dt$ and $z = 3dt$, then $\omega_2(x - z) + \omega_2(z - y) < \omega_2(x - y)$. Therefore, the inequality of Lemma 3 cannot be proved for higher orders.

We will solve this problem with the following result.

Proposition 5. Let $i \in \mathbb{N}_{>1}$, and suppose that

$$d_i(x, y) := |\circ x - \circ y| + \sum_{j=1}^i \omega_j(x - y)$$

verifies the triangle inequality. Then d_i and d_ω are equivalent.

Remark 6. In this statement, we mean $\omega_j(r) := 0$ for $r \in \mathbb{R}$ and $\omega_j(x) := 0$ if $j > N$, where N is the number of summands in the decomposition of x .

Proof: It is readily verified that the d_i -topology is finer than the ω -topology. To prove the converse, let us first consider $y \in B_r(x; d_\omega)$, with $r \leq 1$. Then $d_\omega(x, y) = |\circ x - \circ y| + \omega(x - y) < r \leq 1$. Therefore, $\omega(x - y) < 1$ and hence $\omega(x - y) = 0$. This means $x - y \in \mathbb{R}$ and hence $\omega_j(x - y) = 0$ and $d_i(x, y) = d_\omega(x, y) < r$. Therefore, $B_r(x; d_\omega) \subseteq B_r(x; d_i)$ if $r \leq 1$. Finally, for a generic ball $B_s(x; d_i)$, take $n \in \mathbb{N}_{>0}$ such that $\frac{s}{n} \leq 1$, then $B_s(x; d_i) \supseteq B_{s/n}(x; d_i) \supseteq B_{s/n}(x; d_\omega)$. \square

2.1 Fermat and ω -completeness of $\bullet\mathbb{R}$

The proof of the following result follows directly from the equality $d_F(s_n, s_m) = |\circ s_n - \circ s_m|$.

Theorem 7. *With respect to the Fermat metric d_F , the ring $\bullet\mathbb{R}$ is complete. In particular, if $(s_n)_n$ is a Cauchy sequence with respect to d_F , then $(\circ s_n)_n$ is a standard Cauchy sequence of \mathbb{R} . Let $r \in \mathbb{R}$ be its limit, then*

$$\forall x \in \mu(x) : (s_n)_n \text{ converges to } x \text{ with respect to } d_F.$$

Before studying the ω -topology, we want to understand better the intuition underlying this metric, because it is strictly related to nilpotency of every infinitesimal of $\bullet\mathbb{R}$. Let us start with an example:

$$s_n := \frac{1}{n} + dt_{\frac{N}{n}} \quad \forall n \in \mathbb{N}_{>0},$$

where $N \in \mathbb{N}_{>0}$. We have $d_\omega(s_n, 0) = \frac{1}{n} + \omega\left(\frac{1}{N} + dt_{\frac{N}{n}} - 0\right) = \frac{1}{n} + \frac{N}{n} \rightarrow 0$ as $n \rightarrow +\infty$. Therefore, $(s_n)_n$ converges to 0 in the ω -topology. However, exactly because of nilpotency, we also have

$$\forall n > N : dt_{\frac{N}{n}} = 0 \quad \text{hence} \quad s_n = \frac{1}{n} \in \mathbb{R}.$$

More generally, if $d_\omega(s_n, x) \rightarrow 0$, then $|\circ s_n - \circ x| \rightarrow 0$, but also $\omega(s_n - x) \rightarrow 0$. This means that the order $\omega(s_n - x)$ goes through smaller and smaller infinitesimals. However, because of nilpotency, infinitesimals of $\bullet\mathbb{R}$ cannot have order less than 1, and hence the order $\omega(s_n - x)$ must collapse from 1 to 0. The following theorems will clarify this intuition.

Theorem 8. *Let $x \in \bullet\mathbb{R}$ and $s \in (0, 1]_{\mathbb{R}}$, then*

$$B_s(x; d_\omega) = (-s, s)_{\mathbb{R}} + \{x\},$$

that is, every $y \in B_s(x; d_\omega)$ can be written as $y = r + x$, with $r \in \mathbb{R}$, $-s < r < s$.

Proof: For $y \in B_s(x; d_\omega)$ we have $|\circ y - \circ x| < s$ and $\omega(y - x) < s \leq 1$, therefore $\omega(y - x) = 0$ and $y - x =: r \in \mathbb{R}$. Moreover, $r = \circ y - \circ x$ so that $|r| < s$, that is $r \in (-s, s)_{\mathbb{R}}$. Vice versa, if $y \in (-s, s)_{\mathbb{R}} + \{x\}$, then $y = r + x$, with $|r| < s$. So $|\circ y - \circ x| = |r| < s$ and $\omega(y - x) = \omega(r) = 0$ and $d_\omega(y, x) < s$. \square

The following theorem characterizes ω -convergent sequences formalizing the intuition presented above.

Theorem 9. *Let $(s_n)_n$ be a sequence of $\bullet\mathbb{R}$, then we have that*

$$\lim_{n \rightarrow +\infty} s_n = x \in \bullet\mathbb{R} \tag{2.3}$$

with respect to the omega topology if and only if the following conditions hold

1. $\lim_{n \rightarrow +\infty} \circ s_n = \circ x$ in \mathbb{R}
2. *The sequence of infinitesimal parts is eventually constant and equal to δx , i.e.*

$$\exists N \in \mathbb{N} : \forall n \geq N : \delta s_n = \delta x.$$

(where we recall that $\delta y := y - \circ y$.)

Proof: We only have to prove 2, as the rest of the proof is immediate. Because of (2.3), we have that

$$\lim_{n \rightarrow +\infty} \omega(s_n - x) = 0 = \lim_{n \rightarrow +\infty} \omega(\delta s_n - \delta x).$$

Therefore $\omega(\delta s_n - \delta x) < 1$ for $n \geq N$, and hence $\omega(\delta s_n - \delta x) = 0$. This means that $\delta s_n - \delta x \in \mathbb{R}$ and hence $\delta s_n - \delta x = 0$ because $\delta s_n, \delta x \in D_\infty$. \square

Example 10. $s_n = \frac{1}{n} + dt_{\frac{N}{n}} + dt_2 \rightarrow dt_2$, whereas $s_n = dt_{1+\frac{1}{n}}$ is not an ω -convergent sequence.

Theorem 11. *The ring $\bullet\mathbb{R}$ is complete with respect to the metric d_ω .*

Proof: Let $(s_n)_n$ be an ω -Cauchy sequence, then

$$\lim_{\substack{n \rightarrow +\infty \\ m \rightarrow +\infty}} |{}^\circ s_n - {}^\circ s_m| = 0,$$

so that the $\lim_{n \rightarrow +\infty} {}^\circ s_n =: r \in \mathbb{R}$ exists. Moreover, we have

$$\lim_{\substack{n \rightarrow +\infty \\ m \rightarrow +\infty}} \omega(s_m - s_n) = 0 = \lim_{\substack{n \rightarrow +\infty \\ m \rightarrow +\infty}} \omega(\delta s_m - \delta s_n).$$

Like in the previous proof, we have $\delta s_n = \delta s_m = \delta s_N$ for $n, m \geq N$. Therefore, setting $x := r + \delta s_N \in \bullet\mathbb{R}$ we have $\lim_{n \rightarrow +\infty} s_n = x$. \square

2.1.1 The omega metric on D_∞ can be defined by a pseudovaluation

In this section we want to prove that the restriction of d_ω to the set of all the infinitesimals D_∞

$$d_\omega(h, k) = \omega(h - k) \quad \forall h, k \in D_\infty$$

is induced by a pseudovaluation. We will see that the same idea doesn't work outside D_∞ . These notes have also the aim to fix some small error made on the same topic in [13].

For completeness, we start from the definition of pseudovaluation on a generic ring with values in $\mathbb{R} \cup \{+\infty\}$.

Definition 12. Let A be a ring, then we say that $v : A \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a pseudovaluation if the following conditions hold:

1. $v : A \longrightarrow \mathbb{R} \cup \{+\infty\}$
2. $\forall x \in A : v(x) = +\infty \iff x = 0$
3. $v(x \cdot y) \geq v(x) + v(y)$
4. $v(x + y) \geq \min[v(x), v(y)]$
5. $v(x - y) = v(y - x)$

Obviously, the last three conditions are supposed to hold for every $x, y \in A$.

Remark 13.

1. We assume the usual rules about the relationships between $+\infty$ and order or sum on \mathbb{R} : $+\infty + r = +\infty > r$ for every $r \in \mathbb{R}$.
2. The essential difference between a pseudovaluation and a valuation is property β , where an inequality replaces an equality. Indeed, it is not hard to prove that equality cannot hold in any ring with zero divisors, like $\bullet\mathbb{R}$.

To motivate the necessity of our definition of pseudovaluation, we anticipate the following lemma, with which we can treat terms of the form $\omega(x \cdot y)$.

Lemma 14. *Let $x, y \in \bullet\mathbb{R}$, then*

$$\omega(x \cdot y) = \begin{cases} \omega(x) & \text{if } x \in D_\infty, y \notin D_\infty \\ \left[\frac{1}{\omega(x)} + \frac{1}{\omega(y)} \right]^{-1} & \text{if } x, y \in D_\infty \setminus \{0\} \end{cases}$$

Moreover, if $x, y \notin \mathbb{R}$, then

$$\omega(x \cdot y) \leq \omega(x) \cdot \omega(y).$$

Proof: Using the infinitesimal parts, we can write $x \cdot y = {}^\circ x \cdot {}^\circ y + {}^\circ x \cdot \delta y + {}^\circ y \cdot \delta x + \delta x \cdot \delta y$, so that, by the definition of order, the equality $\omega(x \cdot y) = \omega({}^\circ x \cdot \delta y + {}^\circ y \cdot \delta x + \delta x \cdot \delta y)$ follows.

In the last case of the statement, i.e. $x, y \in D_\infty \setminus \{0\}$, we have

$$\begin{aligned} \omega(x \cdot y) &= \omega(\delta x \cdot \delta y) = \left[\frac{1}{\omega(x)} + \frac{1}{\omega(y)} \right]^{-1} = \frac{\omega(x) \cdot \omega(y)}{\omega(x) + \omega(y)} \leq \\ &\leq \frac{1}{2} \omega(x) \cdot \omega(y) \leq \omega(x) \cdot \omega(y) \end{aligned}$$

Moreover, let us also note that $\frac{1}{\omega(x \cdot y)} = \frac{1}{\omega(x)} + \frac{1}{\omega(y)} \geq \frac{1}{\omega(x)}$, and hence

$$\omega(x \cdot y) \leq \omega(x). \quad (2.4)$$

Below we will use this inequality for suitable infinitesimals x and y .

If $x \in D_\infty$ and $y \notin D_\infty$, the cases $\delta x = 0$ or $\delta x \cdot \delta y = 0$ are immediate. Otherwise, we have

$$\begin{aligned} \omega(x \cdot y) &= \omega({}^\circ x \cdot \delta y + {}^\circ y \cdot \delta x + \delta x \cdot \delta y) = \omega({}^\circ y \cdot \delta x + \delta x \cdot \delta y) = \\ &= \max[\omega({}^\circ y \cdot \delta x), \omega(\delta x \cdot \delta y)]. \end{aligned} \quad (2.5)$$

Now, we can apply (2.4) to the product $\delta x \cdot \delta y$, obtaining

$$\omega(\delta x \cdot \delta y) \leq \omega(\delta x) = \omega({}^\circ y \cdot \delta x). \quad (2.6)$$

Note that the last equality is due to the fact that $y \notin D_\infty$, so that ${}^\circ y \neq 0$. Therefore, from (2.5) and (2.6) we obtain $\omega(x \cdot y) = \omega({}^\circ y \cdot \delta x) = \omega(\delta x) = \omega(x)$. Finally, $\omega(x) \leq \omega(x) \cdot \omega(y)$ if $y \notin \mathbb{R}$ because in that case $\omega(y) \geq 1$. \square

Remark 15. In the statement of the previous theorem, the case where $x, y \notin D_\infty$ is not included. This is done to avoid an overcomplicated statement. Indeed, we have several sub-cases:

1. If ${}^\circ x \cdot \delta y + {}^\circ y \cdot \delta x + \delta x \cdot \delta y = 0 \implies \omega(x \cdot y) = 0$
2. If ${}^\circ x \cdot \delta y + {}^\circ y \cdot \delta x = 0$ and $\delta x \cdot \delta y \neq 0 \implies \omega(x \cdot y) = \left[\frac{1}{\omega(x)} + \frac{1}{\omega(y)} \right]^{-1}$
3. If ${}^\circ x \cdot \delta y + {}^\circ y \cdot \delta x \neq 0 \implies \omega(x \cdot y) = \max[\omega(x), \omega(y)]$.

The idea to define a pseudovaluation is to derive the property $v(x \cdot y) \geq v(x) + v(y)$ from $\omega(x \cdot y) \leq \omega(x) \cdot \omega(y)$, so that the natural try is the following

Definition 16. For every infinitesimal $h \in D_\infty$, we define

$$v(h) := -\log[\omega(h)],$$

where we use the convention that $-\log(0) := +\infty$.

Remark 17.

1. The metric associated to v is $e^{-v(h-k)} = \omega(h-k) = d_\omega(h, k)$.
2. If we define $v(x) := -\log[{}^\circ x + \omega(x)]$ for every $x \in {}^\bullet\mathbb{R}$, we do not obtain a pseudovaluation. Indeed, we have e.g. $v(3+5) < \min[v(3), v(5)]$.

Theorem 18. $v : D_\infty \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a pseudovaluation on the subring (ideal) D_∞ of all the infinitesimals.

Proof: We only have to prove property 4 of Definition 12, as the others are immediate. We can suppose $x + y \neq 0$, because otherwise the proof is obvious. Since $x, y \in D_\infty$ and $x + y \neq 0$, we have that $\omega(x + y) = \max[\omega(x), \omega(y)]$. We will proceed in the case $\omega(x) \geq \omega(y)$, the opposite being analogous. Therefore, $\omega(x + y) = \omega(x)$ and $v(x + y) = -\log[\omega(x + y)] = v(x)$. From $\omega(x) \geq \omega(y)$ we have $-\log \omega(x) \leq -\log \omega(y)$, so that $v(x) \leq v(y)$ and $\min[v(x), v(y)] = v(x) = v(x + y)$, which is our conclusion. \square

3 Ideals and their characterization

In this section, we want to study the ring of Fermat reals from the point of view of some standard algebraic structure. To begin with, we note a few elementary algebraic properties of ${}^\bullet\mathbb{R}$:

- There are no nontrivial idempotents in ${}^\bullet\mathbb{R}$.
- ${}^\bullet\mathbb{R}$ is not reduced.
- $x \in {}^\bullet\mathbb{R}$ is a zero divisor iff $x \in D_\infty$ iff x is non-invertible.

- $\bullet\mathbb{R}$ is an exchange ring (i.e., for each $x \in \bullet\mathbb{R}$ there exists an idempotent such that $x + e$ is invertible).
- $\bullet\mathbb{R}$ is an l -ring (lattice ordered ring), as well as a normal f -ring (cf. [4]).

Next we want to study the ideals of $\bullet\mathbb{R}$. We will see that, as opposed to other rings containing infinitesimals (see, e.g., [27]) the ideals in $\bullet\mathbb{R}$ can be exhaustively described. Of course, this is essentially due to the very simple family of little-oh polynomials used as representatives of new numbers in $\bullet\mathbb{R}$. We start by proving that the only maximal ideal is the set D_∞ of all the infinitesimals. The idea to consider only “well behaved” functions $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ (the little-oh polynomials), in the definition of the ring $\bullet\mathbb{R}$, is tied with the fact that Fermat reals do not represent a new foundation for the entire calculus. Indeed, our aim is only to extend ordinary smooth functions, so that it suffices to evaluate them on ‘well-behaved numbers’.

The situation is entirely different in NSA, which aims to be a new, independent foundation of the whole calculus. For example, suppose we want to prove that ordinary continuity of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ at $x_0 \in \mathbb{R}$ is equivalent to

$$\forall x \in {}^*\mathbb{R} : x \simeq x_0 \implies {}^*f(x) \simeq {}^*f(x_0), \quad (3.1)$$

where $x \simeq y$ means that $x - y$ is infinitesimal. However, (3.1) is nothing more than the continuity of the function f stated using sequences, i.e.

$$\forall x \in \mathbb{R}^{\mathbb{N}} : \exists \lim_{n \rightarrow +\infty} x_n = x_0 \implies \exists \lim_{n \rightarrow +\infty} f(x_n) = f(x_0). \quad (3.2)$$

This is equivalent to ordinary continuity only if it is stated for every sequence $x \in \mathbb{R}^{\mathbb{N}}$, as is obvious from the corresponding proof.

Lemma 19. *Let I be a proper ideal of the ring $\bullet\mathbb{R}$, then $I \subseteq D_\infty$.*

Proof: Let $x \in I$ and suppose, by contradiction, that ${}^\circ x \neq 0$, then x would be invertible and for every $y \in \bullet\mathbb{R}$ we could write $y = y \cdot x^{-1} \cdot x$. By hypothesis $x \in I$, which is an ideal, so we would have $y \in I$, that is $I = \bullet\mathbb{R}$ which is impossible because I is proper by hypothesis. \square

Directly from the decomposition of every Fermat real it follows that

$$\bullet\mathbb{R}/D_\infty \simeq \mathbb{R}$$

and hence D_∞ is a maximal ideal. With the following result, we prove that in fact it is the only one.

Theorem 20. *Let I be a proper maximal ideal of the ring $\bullet\mathbb{R}$, then $I = D_\infty$.*

Proof: By the previous lemma, we have that $I \subseteq D_\infty$. The ring $\bullet\mathbb{R}$ is commutative and with unity, so $\bullet\mathbb{R}/I$ is a field because, by hypothesis, I is maximal. Take $x \in D_\infty$. We have two cases: either $x + I = I$ or $x + I \neq I$. In the first one, we have $x + 0 = x \in I$. In the second one, $x + I$ would be invertible in the field $\bullet\mathbb{R}/I$, so, for some $y \in \bullet\mathbb{R}$, we can write

$$(y + I) \cdot (x + I) = 1 + I \quad \text{i.e.} \quad yx + I = 1 + I,$$

that is $yx+i = 1+j$ for some $i, j \in I$. Taking the standard parts in this equality we obtain ${}^\circ y \cdot {}^\circ x + {}^\circ i = 1 + {}^\circ j$. From $I \subseteq D_\infty$ we deduce that ${}^\circ i = {}^\circ j = 0$ and, therefore, that ${}^\circ y \cdot {}^\circ x = 1$, that is ${}^\circ x \neq 0$, which is impossible because $x \in D_\infty$. \square

This proof can be easily generalized to the following

Theorem 21. *Let R, F be two commutative rings with unity, with F non trivial. Moreover, let $s : R \rightarrow F$ be a ring morphism and set $D := \ker(s)$. Finally, let us suppose that*

$$\forall x \in R : x \notin D \implies x \text{ is invertible in } R.$$

Then D is the only maximal ideal of R .

In our case $s = {}^\circ(-)$ is the standard part map.

Finally, we prove that every proper ideal is either of the form

$$D_a = \{x \in D_\infty \mid \omega(x) < a + 1\} \quad \text{for some } a \in \mathbb{R}_{\geq 1} \cup \{\infty\},$$

or of the form

$$I_a := \{x \in D_\infty \mid \omega(x) \leq a\} \quad \text{for some } a \in \mathbb{R}_{\geq 1}.$$

As we mentioned above, the first type of ideal is used in infinitesimal Taylor formulas, whereas the second is used in the study of the equivalence relation $=_a$ of equality up to a -th order infinitesimals.

This characterization is tied to the possibility to solve in ${}^\bullet\mathbb{R}$ the following class of linear equations.

Theorem 22. *If $a, b, c \in {}^\bullet\mathbb{R}$ and $a < c < a + b$, then*

$$\exists x \in {}^\bullet\mathbb{R} : a + x \cdot b = c$$

For the proof of this theorem, see [10, 9]. Let us note that we cannot have uniqueness of solutions, due to nilpotency. For example, if $a = 0$, $c = dt_2 + dt$ and $b = dt_3$, then $x = dt_6 + dt_{3/2}$ is a solution of $a + x \cdot b = c$, but $x + dt$ is another solution. Moreover, let us note that this theorem is not in contradiction with the non Archimedean property of ${}^\bullet\mathbb{R}$ (let $a = 0$ and $b \in D_\infty$) because of the inequalities that c must verify for a solution to exist.

Using this result, we can prove the desired characterization:

Theorem 23. *Let J be a proper ideal of ${}^\bullet\mathbb{R}$, and $b \in J$. Moreover, set*

$$O(J) := \{\omega(j) \in \mathbb{R}_{\geq 0} \mid j \in J\} \quad , \quad a := \sup O(J).$$

Then

1. $\forall c \in {}^\bullet\mathbb{R} : -b < c < b \implies c \in J$
2. $a \in O(J) \implies J = I_a$

$$3. a \notin O(J) \implies a \geq 1 \text{ and } J = D_{a-1}.$$

Proof: To prove 1, set $a = 0$ in Theorem 22. Then since $-b < c < b$, we can distinguish two cases (we recall that the order relation in $\bullet\mathbb{R}$ is total). If $c \geq 0$, then $0 \leq c < b$ and we can hence solve the equation $x \cdot b = c$ and, therefore, $c \in J$ because $b \in J$. Otherwise, $c < 0$ and so $0 < -c < b$. We solve the equation $x \cdot b = -c$, that is $(-x) \cdot b = c$ so that $c \in J$ again.

To prove 2 we first note that $J \subseteq I_a$ by the definition of a . Vice versa, let $i \in I_a$, i.e. $\omega(i) \leq a$. By hypothesis, $a \in O(J)$, so that we can write $a = \omega(j)$ for some $j \in J$. We can suppose $j > 0$ because, otherwise, $0 < -j \in J$ and $a = \omega(j) = \omega(-j)$. We distinguish two cases. If $\omega(i) < a = \omega(j)$, then $i < j$ by the properties of the order relation we mentioned in the introduction (see Theorem 4.2.6 in [9]). On the other hand, we also have that $\omega(-i) = \omega(i) < \omega(j)$ and hence $-i < j$ because $j > 0$. Therefore, $-j < i < j$, and the conclusion $i \in J$ follows from 1. Let us note that, in general, we have just proved that

$$\forall i, j \in \bullet\mathbb{R} : j > 0, \omega(i) < \omega(j) \implies -j < i < j. \quad (3.3)$$

In the second case, we suppose that $\omega(i) = a$ so that, by the decompositions of i, j and for suitable $\alpha, \beta \in \mathbb{R}_{\neq 0}$ and $h, k \in D_\infty$, we can write

$$j = \alpha \cdot dt_a + h, \quad \omega(h) < a \quad (3.4)$$

$$i = \beta \cdot dt_a + k, \quad \omega(k) < a. \quad (3.5)$$

Therefore, from (3.4) and (3.3) it follows that $-j < h < j$ and hence $h \in J$ from property 1. So $\alpha \cdot dt_a = j - h \in J$ and $dt_a \in J$ because $\alpha \neq 0$. Hence, we also have that $\beta \cdot dt_a \in J$. Finally, $\omega(k) < a = \omega(j)$ so that $-j < k < j$ from (3.3) and $k \in J$ from property 1. We have proved that $\beta \cdot dt_a, k \in J$, so $i = \beta \cdot dt_a + k \in J$, which is our conclusion.

Finally, to prove 3 we first note that

$$D_{a-1} = \{x \in D_\infty \mid \omega(x) < a\},$$

where we use the conventions $\infty \pm 1 = \infty$. If $a \notin O(J)$ then we have $\omega(j) < a$ for every $j \in J$, and therefore $J \subseteq D_{a-1}$, considering also that $J \subseteq D_\infty$.

Vice versa, if $i \in D_{a-1}$, then we can find $j \in J$ such that $\omega(i) < \omega(j) < a$ because $a = \sup O(J)$. As usual, we can suppose $j > 0$. From (3.3), it follows $-j < i < j$ and hence $i \in J$ by property 1. To finish, let us note that because $\omega(j) \geq 1$ or $\omega(j) = 0$ we necessarily have that $a = 0$ or $a \geq 1$. However, the first possibility would imply $J = \{0\}$ and hence $a \in O(J)$, which is impossible by hypothesis. \square

4 Roots of infinitesimals

In the ring of Fermat reals $\bullet\mathbb{R}$, the existence of non zero nilsquare elements:

$$h \neq 0 \quad , \quad h^2 = 0, \quad (4.1)$$

is incompatible with the existence of a square root and of an absolute value with the usual properties. In other words, if we want to define roots of infinitesimals, we have to avoid from (4.1) the following inference:

$$\begin{aligned} h^2 = 0 \quad \text{therefore} \quad \sqrt{h^2} = \sqrt{0} = 0 \\ \sqrt{h^2} = |h| = 0 \quad \text{therefore} \quad h = 0. \end{aligned}$$

We recall that only smooth functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ can be extended to $\bullet\mathbb{R}$. In particular:

- Because they are locally Lipschitz, these functions verify

$$\forall x, y \in \bullet\mathbb{R} : x = y \text{ in } \bullet\mathbb{R} \implies f \circ x = f \circ y \text{ in } \bullet\mathbb{R}.$$

- Because they are smooth, they take little-oh polynomials into little-oh polynomials:

$$\forall x \in \mathbb{R}_o[t] : f \circ x \in \mathbb{R}_o[t].$$

It is hence natural to expect some problems extending, e.g., the square root to the whole of $\bullet\mathbb{R}$.

The first natural solution is to extend the roots only where they are smooth, i.e. on $\mathbb{R}_{\neq 0}$. This is equivalent to defining the roots only for invertible Fermat reals (and positive in case of even roots or irrational powers). For details about this approach, see [9], section 4.3, or [11], section 12.

Another problem we have to take into account, and concerning roots of infinitesimals, is that the equation $x^2 = c$, for $c \in D_\infty$, always has infinitely many solutions, e.g.

$$\begin{aligned} (dt_4)^2 &= dt_{4/2} = dt_2 \\ (dt_4 + h)^2 &= dt_2 + h^2 + 2h dt_4 = dt_2 \quad \forall h \in D_\infty : \omega(h) < \frac{4}{3}. \end{aligned}$$

Therefore, we have infinitely many square roots of an infinitesimal. This means that, although in \mathbb{R} we have that $(-)^2 : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is bijective and $\sqrt{} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is its (left and right) inverse, in $\bullet\mathbb{R}_{\geq 0}$ we don't have injectivity, and, therefore, we can have, at most, a right inverse. Indeed, we will prove that $(-)^2 : \bullet\mathbb{R}_{\geq 0} \longrightarrow \bullet\mathbb{R}_{\geq 0}$ is surjective, and

$$\forall x \in \bullet\mathbb{R}_{\geq 0} : (\sqrt{x})^2 = x \quad \text{but} \quad \exists k : \sqrt{x^2} =_k x.$$

Because we have infinitely many solutions of equations of the type $x^p = c$, $p \in \mathbb{R}_{>1}$, a first idea is to choose, among them, the simplest solution. Here, with “simplest”, we mean “the solution x without unnecessary terms in its decomposition, i.e. without terms that become zero taking the power x^p ”. A similar

idea of “simplest solution” has already been used in [9] to define derivatives of smooth functions defined on infinitesimal sets. For example, both $x = dt_4$ and $x = dt_4 + dt$ are solutions of the equation $x^2 = dt_2$, but, intuitively, the first one is simpler compared to the second one, which contains the unnecessary term dt .

However, there is another, more manageable idea to define roots of infinitesimal numbers. Let

$$c = \sum_{i=1}^{N_c} {}^\circ c_i dt_{\omega_i(c)}$$

be the decomposition of $c \in \mathcal{D}_\infty$. Suppose $c > 0$, so that $\omega(c) = \omega_1(c) > \omega_2(c) > \dots > \omega_{N_c}(c) \geq 1$ and ${}^\circ c_i \neq 0$, $c_1 > 0$, then, for $p \in \mathbb{R}_{>0}$, we would like to write

$$\begin{aligned} c^p &= \left(\sum_{i=1}^{N_c} {}^\circ c_i dt_{\omega_i(c)} \right)^p = \\ &= \left[{}^\circ c_1 dt_{\omega(c)} \cdot \left(1 + \sum_{i=2}^{N_c} \frac{{}^\circ c_i}{{}^\circ c_1} dt_{\omega_i(c) \ominus \omega(c)} \right) \right]^p = \\ &= ({}^\circ c_1)^p dt_{\frac{\omega(c)}{p}} \cdot \left(1 + \sum_{i=2}^{N_c} \frac{{}^\circ c_i}{{}^\circ c_1} dt_{\omega_i(c) \ominus \omega(c)} \right)^p, \end{aligned} \quad (4.2)$$

where

$$\frac{1}{0} := \infty, \quad a \oplus b := \left(\frac{1}{a} + \frac{1}{b} \right)^{-1}, \quad a \ominus b := \left(\frac{1}{a} - \frac{1}{b} \right)^{-1} \quad \forall a, b \in \mathbb{R}.$$

However, the right hand side of (4.2) is now a well defined term, because the base of the p -th power is invertible.

Remark 24.

1. Note that the right hand side of (4.2) is well defined if ${}^\circ c_1 \neq 0$, i.e. if $c \in D_\infty \setminus \{0\}$, and because $\omega(c) > \omega_i(c)$, so that $\omega_i(c) \ominus \omega(c) > 0$ and hence $dt_{\omega_i(c) \ominus \omega(c)}$ is well defined. Moreover, it is not hard to prove that $\omega_i(c) \ominus \omega(c) > 1$ if $i > 1$.
2. It can be useful to note that setting

$$\begin{aligned} \ominus b &:= -b \\ a \odot b &:= a \cdot b, \end{aligned}$$

we easily have that $(\mathbb{R}_\infty, \oplus, \ominus, \odot, \infty)$ is a ring and the reciprocal function $\frac{1}{(-)} : \mathbb{R}_\infty \longrightarrow \mathbb{R}$ is a ring isomorphism.

Definition 25. Let $c \in D_\infty$, $c > 0$, and $p \in \mathbb{R}_{>0}$, then

$$c^p := ({}^\circ c_1)^p dt_{\frac{\omega(c)}{p}} \cdot \left(1 + \sum_{i=2}^{N_c} \frac{{}^\circ c_i}{{}^\circ c_1} dt_{\omega_i(c) \ominus \omega(c)} \right)^p.$$

Of course, if $p = \frac{m}{n}$, where $m, n \in \mathbb{N}$ and n is odd, the hypothesis $c > 0$ can be dropped.

Example 26.

1. Let us find \sqrt{dt} using the previous definition. In this case, we have $N_c = 1$, ${}^\circ c_1 = 1$, $\omega(c) = 1$, so

$$\sqrt{dt} = (1)^{1/2} dt_{1/2} \cdot (1 + 0)^{1/2} = dt_2.$$

2. We want to find $\sqrt{dt_2 + dt}$:

$$\begin{aligned} \sqrt{dt_2 + dt} &= dt_4 \cdot (1 + dt_{1 \ominus 2})^{1/2} = dt_4 (1 + dt_2)^{1/2} = \\ &= dt_4 \cdot \left(1 + \sum_{n=1}^2 \binom{\frac{1}{2}}{n} dt_{2/n} \right) = \\ &= dt_4 \cdot \left(1 + \frac{1}{2} \cdot dt_2 - \frac{1}{8} \cdot dt \right) = \\ &= dt_4 + \frac{1}{2} dt_{4/3}. \end{aligned}$$

We recall that

$$(1 + h)^p = 1 + \sum_{n=1}^{+\infty} \binom{p}{n} \cdot h^n \quad \forall h \in D_\infty \quad (4.3)$$

because of the elementary transfer theorem (Theorem 24 in [13]). Finally, let us note that the series in (4.3) is really a finite sum because of nilpotency of every infinitesimal $h \in D_\infty$.

As expected, we have that $(dt_4 + \frac{1}{2} dt_{4/3})^2 = dt_2 + dt$.

3. $\sqrt{dt_2 - dt_{3/2} - dt} = dt_4 - \frac{1}{2} dt_{12/5} - \frac{1}{8} dt_{12/7} - \frac{9}{16} dt_{4/3} - \frac{37}{128} dt_{12/11}.$

Generalizing these examples, we have that

$$c^p = ({}^\circ c_1)^p dt_{\frac{\omega(c)}{p}} \cdot \left[1 + \sum_{n=1}^{+\infty} \binom{p}{n} \cdot \left(\sum_{i=2}^{N_c} \frac{{}^\circ c_i}{{}^\circ c_1} dt_{\omega_i(c) \ominus \omega(c)} \right)^n \right].$$

In the following theorems, in considering x^p for generic $p \in \mathbb{R}_{>0}$, we will always suppose $x > 0$. However, this hypothesis can be dropped in case of odd roots, and the proofs will remain essentially the same.

Theorem 27. *Let $x \in D_\infty$, $x > 0$, and $p \in \mathbb{R}$, with $0 < p < 1$, then we have:*

1. $(x^p)^{\frac{1}{p}} = x$
2. If $x^{\frac{1}{p}} \neq 0$ and $k := \max \left\{ \omega_2(x), \omega_2 \left[\left(x^{\frac{1}{p}} \right)^p \right] \right\}$, then $\left(x^{\frac{1}{p}} \right)^p =_k x$.

Remark 28.

1. To understand better, it can be useful to clarify what is the difference between the computation of $(x^p)^{1/p}$ and that of $(x^{1/p})^p$:
 - (a) $(x^p)^{1/p}$: Because $p \in [0, 1]_{\mathbb{R}}$, the computation of x^p is included in the Definition 25. Therefore, we must:
 - i. Express x using its decomposition.
 - ii. Use Definition 25.
 - iii. With the obtained result, we finally have to compute the subsequent power $(-)^{1/p}$. However, $\frac{1}{p} > 1$, so that this operation is smooth and doesn't present any problem.
 - (b) $(x^{1/p})^p$: In this case, the situation is the opposite one.
 - i. $\frac{1}{p} > 1$, so the operation $x^{1/p} =: y$ is smooth.
 - ii. However, to compute y^p , we must apply Definition 25, so we firstly need the decomposition of $y = x^{1/p}$. Of course, it is not easy to find this decomposition as a manageable function of the decomposition of x .
2. Let us note that if $x, y \in D_\infty$ and $k := \max [\omega(x), \omega(y)]$, then it is trivially true that $x =_k y$, because $\omega(x-y) = k$ if $x \neq y$, and $\omega(x-y) = 0$ otherwise. This shows that 2 of Theorem 27 is not trivial.
3. Theorem 27 represents an overcoming of the incompatibility between nilpotent infinitesimals and existence of roots. Indeed, property 2 can be applied only if $x^{1/p} \neq 0$. Moreover, if $h \in D_\infty \setminus \{0\}$ and $h^2 = 0$, we have $\sqrt{h^2} = \sqrt{0} = 0$, but, in general, $\sqrt{h^2} \neq |h|$, e.g. $\sqrt{(dt)^2} = 0 \neq |dt| = dt$.

Proof of Theorem 27: Let $x = \sum_{j=1}^N b_j dt_{\beta_j}$ be the decomposition of x . Because $0 < p < 1$, from Definition 25, we have

$$x^p = b_1^p dt_{\frac{\beta_1}{p}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^p.$$

Now, we have to apply the power $(-)^{1/p}$, which is smooth and has the usual properties of powers (see, e.g., [9], section 4.3). Therefore, we can write

$$\begin{aligned}
(x^p)^{\frac{1}{p}} &= \left(b_1^p dt_{\frac{\beta_1}{p}} \right)^{\frac{1}{p}} \cdot \left[\left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^p \right]^{\frac{1}{p}} = \\
&= b_1 dt_{\beta_1} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right) = x.
\end{aligned}$$

This proves 1.

To prove 2, we firstly have to compute the smooth power $(-)^{1/p}$

$$x^{\frac{1}{p}} = \left(\sum_{j=1}^N b_j dt_{\beta_j} \right)^{\frac{1}{p}}.$$

The idea is to use the usual properties of $(-)^{1/p}$ and to gather up the leading term $b_1 dt_{\beta_1}$:

$$\begin{aligned}
x^{\frac{1}{p}} &= \left[b_1 dt_{\beta_1} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right) \right]^{\frac{1}{p}} = \\
&= b_1^{\frac{1}{p}} dt_{p\beta_1} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^{\frac{1}{p}}. \tag{4.4}
\end{aligned}$$

We are not able to find the decomposition of this number, but we can surely claim that

$$\circ \left(x^{\frac{1}{p}} \right)_1 = b_1^{\frac{1}{p}} = (\circ x_1)^{\frac{1}{p}} \tag{4.5}$$

$$\omega \left(x^{\frac{1}{p}} \right) = p \cdot \beta_1 = p \cdot \omega(x) \quad \text{if } x^{\frac{1}{p}} \neq 0 \text{ i.e. if } p \cdot \beta_1 \geq 1. \tag{4.6}$$

This guess is based on the idea that in (4.4), the infinitesimal $b_1^{\frac{1}{p}} \cdot dt_{p\beta_1}$ is multiplied by an invertible number, whose standard part is 1. Indeed, we have the following

Lemma 29. *Let $k, h \in D_\infty$ and $y \in \bullet \mathbb{R}$ such that y is invertible and $k = h \cdot y$. Then*

$$\begin{aligned}
\omega(k) &= \omega(h) \\
\circ k_1 &= \circ h_1 \cdot \circ y.
\end{aligned}$$

We postpone the proof of this lemma to the end of the current proof.
Using (4.5) and (4.6) and Definition 25, we can write

$$\left(x^{\frac{1}{p}}\right)^p = {}^\circ x_1 \, dt_{\omega(x)} \cdot \left(1 + \sum_{i=2}^M \frac{{}^\circ \left(x^{\frac{1}{p}}\right)_i}{{}^\circ x_1} dt_{\omega_i \left(x^{\frac{1}{p}}\right) \ominus \omega(x)}\right)^p, \quad (4.7)$$

where M is the number of terms in the decomposition of $x^{1/p}$ (of which, we really know only the first term). Using again Lemma 29 applied to (4.7), we have

$$\begin{aligned} {}^\circ \left[\left(x^{\frac{1}{p}}\right)^p\right]_1 &= {}^\circ x_1 \\ \omega \left[\left(x^{\frac{1}{p}}\right)^p\right] &= \omega(x). \end{aligned}$$

Let us observe that $x > 0$, hence $x \neq 0$ and $\omega(x) \geq 1$, so that $\omega({}^\circ x_1 \, dt_{\omega(x)}) = \omega(x)$. We have hence proved that the first terms in the decompositions of both x and $\left(x^{\frac{1}{p}}\right)^p$ are the same. Therefore

$$\omega \left[x - \left(x^{\frac{1}{p}}\right)^p\right] \leq \max \left\{ \omega_2(x), \omega_2 \left[\left(x^{\frac{1}{p}}\right)^p\right] \right\} = k$$

and hence $x =_k \left(x^{\frac{1}{p}}\right)^p$. □

Proof of Lemma 29: Because y is invertible, we have that ${}^\circ y \neq 0$. Write the product $k = h \cdot y$ using decompositions

$$\begin{aligned} k &= \sum_{i=1}^{N_k} {}^\circ k_i \, dt_{\omega_i(k)} = \left(\sum_{p=1}^{N_h} {}^\circ h_p \, dt_{\omega_p(h)} \right) \cdot \left({}^\circ y + \sum_{q=1}^{N_y} {}^\circ y_q \, dt_{\omega_q(y)} \right) = \\ &= {}^\circ h_1 {}^\circ y \, dt_{\omega(h)} + \sum_{p=2}^{N_h} {}^\circ h_p {}^\circ y \, dt_{\omega_p(h)} + \sum_{p=1}^{N_h} \sum_{q=1}^{N_y} {}^\circ h_p {}^\circ y_q \, dt_{\omega_p(h) \oplus \omega_q(y)}. \end{aligned} \quad (4.8)$$

However, in general, we have $a \oplus b < \min(a, b)$ for every $a, b \in \mathbb{R}_{>0}$, so that, in (4.8) the leading term is ${}^\circ h_1 {}^\circ y \, dt_{\omega(h)}$ and hence from the uniqueness of decomposition, the conclusion follows. □

Remark 30.

1. Let us observe that, in the hypothesis of Theorem 27, we also have

$$y = x^p \implies \left(y^{\frac{1}{p}}\right)^p = y.$$

In fact, $y^{\frac{1}{p}} = (x^p)^{\frac{1}{p}} = x$, and hence $\left(y^{\frac{1}{p}}\right)^p = x^p = y$. Therefore, for numbers of the form $y = x^p$, the equality 2 of Theorem 27 becomes exact.

We can interpret this result saying that our Definition 25 of c^p gives exactly the simplest solution of the equation $x^{\frac{1}{p}} = c$. Indeed, like in the case $\sqrt{(dt_2 + dt)^2} =_1 dt_2 + dt$, we can say that in \mathcal{L} we don't have an exact equality if the number x contains unnecessary infinitesimals with respect to the power $(-)^{1/p}$, like dt in the previous example. See section 4.3 for a formalization of the notion of “unnecessary term with respect to the power $(-)^{1/p}$ ”.

2. The equality \mathcal{L} , up to infinitesimals, implies that

$$\sqrt{-} : \bullet\mathbb{R}_{\geq 0} \longrightarrow \bullet\mathbb{R}_{\geq 0} \quad \text{is not surjective.}$$

For example $y = dt_2 + dt$ cannot be written as the square root of some number x . Otherwise, we would have

$$y = \sqrt{x} \implies y^2 = (dt_2 + dt)^2 = dt = (\sqrt{x})^2 = x,$$

but then $\sqrt{x} = \sqrt{dt} = dt_2 \neq y$. Of course, this corresponds to saying that the square is not the right inverse of the square root.

3. Trivially, we can consider a smooth function $g : \mathbb{R} \longrightarrow \mathbb{R}$ having a root of order $n \in \mathbb{N}_{>0}$ at $x = 0$, i.e. such that

$$g(h) = a \cdot h^n \quad \forall h \in D_n,$$

where $a \in \mathbb{R}_{\neq 0}$. We can hence define a sort of infinitesimal right inverse of g , setting

$$f(h) := \sqrt[n]{\frac{h}{a}} \quad \forall h \in D_\infty$$

if n is odd or n is even and $a > 0$, and

$$f(h) := \sqrt[n]{\frac{h}{-a}} \quad \forall h \in D_\infty$$

if n is even and $a < 0$. Then we have $g(f(h)) = \pm h$ for every $h \in D_\infty$, with the positive sign in the first case.

4.1 A formula to compute a root

By definition, if $c \in D_\infty \setminus \{0\}$, we have

$$c^p := ({}^\circ c_1)^p dt_{\frac{\omega(c)}{p}} \cdot \left(1 + \sum_{i=2}^{N_c} \frac{{}^\circ c_i}{{}^\circ c_1} dt_{\omega_i(c) \ominus \omega(c)} \right)^p.$$

The p -th power of the invertible term can be computed in several, obviously equivalent, ways.

1. Using the infinitesimal Taylor formula of the function $(1+x)^p$, with

$$x = \sum_{i=2}^{N_c} \frac{{}^\circ c_i}{{}^\circ c_1} dt_{\omega_i(c) \ominus \omega(c)} \in D_{\omega_2(c) \ominus \omega_1(c)}. \quad (4.9)$$

2. Equivalently, we can use the formula $(1+x)^p = 1 + \sum_{n=1}^{+\infty} \binom{p}{n} \cdot x^n$, for $|x| < 1$, which transfers to D_∞ by Theorem 24 of [13].

Applying the second method, we get

$$\begin{aligned} c^p &= ({}^\circ c_1)^p dt_{\frac{\omega(c)}{p}} \cdot \left[1 + \sum_{n=1}^{+\infty} \binom{p}{n} \cdot \left(\sum_{i=2}^{N_c} \frac{{}^\circ c_i}{{}^\circ c_1} dt_{\omega_i(c) \ominus \omega(c)} \right)^n \right] = \\ &= ({}^\circ c_1)^p dt_{\frac{\omega(c)}{p}} \cdot \left[1 + \sum_{n=1}^{k_{c,p}} \binom{p}{n} \sum_{\substack{\gamma \in \mathbb{N}^{N_c-1} \\ |\gamma|=n}} \frac{n!}{\gamma!} \cdot \prod_{i=2}^{N_c} \left(\frac{{}^\circ c_i}{{}^\circ c_1} \right)^{\gamma_{i-1}} dt_{\frac{\omega_i(c) \ominus \omega(c)}{\gamma_{i-1}}} \right], \end{aligned}$$

where

$$k_{c,p} := \begin{cases} \lceil \omega_2(c) - \omega(c) \rceil & \text{if } p \notin \mathbb{N} \\ \min(\lceil \omega_2(c) - \omega(c) \rceil, p) & \text{if } p \in \mathbb{N} \end{cases} \quad (4.10)$$

with $\lceil a \rceil \in \mathbb{N}$ the ceiling of $a \in \mathbb{R}$, that is the smallest integer greater than or equal to a . Note that the first alternative of (4.10) is due to (4.9), whereas the second one is also due to the equality $\binom{p}{n} = 0$ if $n > p$.

Using Theorem 13 of [13], we have

$$\begin{aligned} \prod_{i=2}^{N_c} dt_{\frac{\omega_i(c) \ominus \omega(c)}{\gamma_{i-1}}} \neq 0 &\iff \sum_{i=2}^{N_c} \frac{\gamma_{i-1}}{\omega_i(c) \ominus \omega(c)} \leq 1 \\ &\iff \bigoplus_{i=2}^{N_c} \frac{\omega_i(c) \ominus \omega(c)}{\gamma_{i-1}} \geq 1. \end{aligned}$$

In the following, we will set $\omega(c, \gamma) := \bigoplus_{i=2}^{N_c} \frac{\omega_i(c) \ominus \omega(c)}{\gamma_{i-1}}$, so that

$$\prod_{i=2}^{N_c} dt_{\frac{\omega_i(c) \ominus \omega(c)}{\gamma_{i-1}}} = dt_{\omega(c, \gamma)}.$$

Finally, we obtain the formula

$$c^p = ({}^\circ c_1)^p dt_{\frac{\omega(c)}{p}} + \sum_{n=1}^{k_{c,p}} \binom{p}{n} \sum_{\gamma \in \Gamma_{c,n}} \frac{n!}{\gamma!} \cdot {}^\circ c_1^{p-n} \cdot {}^\circ c_2^{\gamma_1} \cdot \dots \cdot {}^\circ c_{N_c}^{\gamma_{N_c-1}} dt_{\omega(c, \gamma) \oplus \frac{\omega(c)}{p}}, \quad (4.11)$$

where

$$\Gamma_{c,n} := \{\gamma \in \mathbb{N}^{N_c-1} \mid |\gamma| = n, \omega(c, \gamma) \geq 1\}.$$

4.2 Properties of roots: the general theorem

For generic $x, y \in \bullet\mathbb{R}$ we can state the following

Theorem 31. *Let $x, y \in \mathcal{D}_\infty$ be strictly positive infinitesimals, and $p, q \in \mathbb{R}_{>0}$, then:*

1. $\omega[(x^p)^q] = \omega(x^{pq}) =: o_1$ and $^\circ[(x^p)^q]_1 = ^\circ(x^{pq})_1$.
2. $\omega[(x \cdot y)^p] = \omega(x^p \cdot y^p) =: o_2$ and $^\circ[(x \cdot y)^p]_1 = ^\circ(x^p \cdot y^p)_1$.
3. $\exists k \in \mathbb{R} : 1 \leq k < o_1$ and $(x^p)^q =_k x^{pq}$.
4. $\exists k \in \mathbb{R} : 1 \leq k < o_2$ and $(x \cdot y)^p =_k x^p \cdot y^p$.
5. $x^p \cdot y^q = x^{p+q}$.

Before proving this theorem, we need the following very useful lemma:

Lemma 32. *Let $c = \sum_{j=1}^M a_j dt_{\alpha_j}$, with $\alpha_1 > \alpha_j \geq 1$ for every $j = 2, \dots, M$, and $a_1 > 0$. Let us note explicitly that not necessarily this is the decomposition of c . Then*

$$c^p = a_1^p dt_{\frac{\alpha_1}{p}} \cdot \left(1 + \sum_{j=2}^M \frac{a_j}{a_1} dt_{\alpha_j \ominus \alpha_1} \right)^p.$$

Of course, this lemma states that the formula used for the definition of c^p can also be used starting from a representation $c = \sum_{j=1}^M a_j dt_{\alpha_j}$ which is not necessarily the decomposition of c . To apply this lemma, the important step is to find the greatest infinitesimal $a_1 dt_{\alpha_1}$ and to check that all the other terms $a_j dt_{\alpha_j}$ can be zero only if $a_j = 0$.

Proof of Lemma 32: Starting from $c = \sum_{j=1}^M a_j dt_{\alpha_j}$, we firstly sum all the coefficients a_j having the same infinitesimal dt_{α_j} , i.e. if

$$\begin{aligned} \bar{a}_q &:= \sum \{a_j \mid j = 1, \dots, M, \alpha_j = q\} \quad \forall q \in \{\alpha_j \mid j = 1, \dots, M\} \\ O &:= \{\alpha_j \mid j = 1, \dots, M, \bar{a}_{\alpha_j} \neq 0\} =: \{q_1, \dots, q_N\}, \end{aligned} \quad (4.12)$$

then

$$c = \sum_{q \in O} \bar{a}_q dt_q = \sum_{i=1}^N \bar{a}_{q_i} dt_{q_i}. \quad (4.13)$$

Let us note that in (4.12), q_1, \dots, q_N is any enumeration of the elements of the set of all orders O . Now, all the summands in (4.13) are non zero, because of our definition of the set O . Therefore, reordering the summands in (4.13),

we obtain the decomposition of c . Formally, this means that we can find a permutation σ of $\{1, \dots, N\}$ such that

$$c = \sum_{i=1}^N \bar{a}_{q_{\sigma_i}} dt_{q_{\sigma_i}} \quad (4.14)$$

is the decomposition of c . Let us note that, to obtain (4.14), we need that for every $i = 1, \dots, N$ we can find $j = 1, \dots, M$ such that $q_i = \alpha_j \geq 1$. By definition of decomposition, q_{σ_1} is the maximum order in (4.14), i.e. $q_{\sigma_1} = \max\{q_1, \dots, q_N\} = \max\{\alpha_j \mid \alpha_j \in O\}$. However, we have that $q_{\sigma_1} = \alpha_1$, because $\alpha_1 > \alpha_j$ by hypothesis, and because

$$\bar{a}_{\alpha_1} = \sum \{a_j \mid j = 1, \dots, M, \alpha_j = \alpha_1\} = a_1 \neq 0,$$

so that $\alpha_1 \in O$ and so $\bar{a}_{q_{\sigma_1}} = \bar{a}_{\alpha_1} = a_1$. We can now apply our Definition 25 using the decomposition (4.14):

$$c^p = a_1^p dt_{\frac{\alpha_1}{p}} \cdot \left(1 + \sum_{i=2}^N \frac{\bar{a}_{q_{\sigma_i}}}{a_1} dt_{q_{\sigma_i} \ominus \alpha_1} \right)^p.$$

Now, we only have to retrace the previous steps, so as to eliminate σ , q , \bar{a} , etc.

$$\begin{aligned} c^p &= a_1^p dt_{\frac{\alpha_1}{p}} \cdot \left(1 + \sum_{i=2}^N \frac{\bar{a}_{q_i}}{a_1} dt_{q_i \ominus \alpha_1} \right)^p = \\ &= a_1^p dt_{\frac{\alpha_1}{p}} \cdot \left(1 + \sum_{\substack{q \in O \\ q \neq \alpha_1}} \frac{\bar{a}_q}{a_1} dt_{q \ominus \alpha_1} \right)^p = \\ &= a_1^p dt_{\frac{\alpha_1}{p}} \cdot \left(1 + \sum_{\substack{j=2 \\ \bar{a}_{\alpha_j} \neq 0}}^M \frac{a_j}{a_1} dt_{\alpha_j \ominus \alpha_1} + \sum_{\substack{j=2 \\ \bar{a}_{\alpha_j} = 0}}^M \frac{0}{a_1} dt_{\alpha_j \ominus \alpha_1} \right)^p = \\ &= a_1^p dt_{\frac{\alpha_1}{p}} \cdot \left(1 + \sum_{i=2}^N \frac{a_j}{a_1} dt_{\alpha_j \ominus \alpha_1} \right)^p, \end{aligned}$$

which is our conclusion. \square

Proof of Theorem 31: To prove 1, let $x = \sum_{j=1}^N b_j dt_{\beta_j}$ be the decomposition of x . The idea is to use formula (4.11) to compute x^p , and then Lemma 32 to compute $(x^p)^q$. To avoid heavy notations, we will use the simplified symbols $k := k_{x,p}$ and $\Gamma := \Gamma_{x,n}$:

$$(x^p)^q = \left[b_1^p dt_{\frac{\beta_1}{p}} + \sum_{n=1}^k \binom{p}{n} \sum_{\gamma \in \Gamma} \frac{n!}{\gamma!} \cdot b_1^{p-1} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_{N_c}^{\gamma_{N_c-1}} dt_{\omega(x, \gamma) \oplus \frac{\beta_1}{p}} \right]^q.$$

Before using Lemma 32, we need to prove that $dt_{\frac{\beta_1}{p}}$ is the greatest infinitesimal, so let us compute

$$\begin{aligned} \left[\omega(x, \gamma) \oplus \frac{\beta_1}{p} \right]^{-1} &= \frac{p}{\beta_1} + \sum_{i=2}^N \frac{\gamma_{i-1}}{\beta_i \ominus \beta_1} = \\ &= \frac{p}{\beta_1} + \sum_{i=2}^N \gamma_{i-1} \left(\frac{1}{\beta_i} - \frac{1}{\beta_1} \right) = \\ &= \frac{p-n}{\beta_1} + \sum_{i=2}^N \frac{\gamma_{i-1}}{\beta_i}. \end{aligned}$$

So, we need to prove that $\frac{p}{\beta_1} < \frac{p-n}{\beta_1} + \sum_{i=2}^N \frac{\gamma_{i-1}}{\beta_i}$, that is $n < \sum_{i=2}^N \gamma_{i-1} \cdot \frac{\beta_1}{\beta_i}$. In fact, $\beta_1 > \beta_i$ so that $\sum_{i=2}^N \gamma_{i-1} \cdot \frac{\beta_1}{\beta_i} > \sum_{i=2}^N \gamma_{i-1} = n$. Moreover, we can suppose to restrict the set Γ to those γ such that $\omega(x, \gamma) \oplus \frac{\beta_1}{p} \geq 1$, because, otherwise, the corresponding term $dt_{\omega(x, \gamma) \oplus \frac{\beta_1}{p}} = 0$. We can hence apply the Lemma 32, obtaining

$$(x^p)^q = b_1^{pq} dt_{\frac{\beta_1}{pq}} \cdot \left[1 + \sum_{n=1}^k \binom{p}{n} \sum_{\gamma \in \Gamma} \frac{n!}{\gamma!} \cdot b_1^{p-n} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_{N_c}^{\gamma_{N_c-1}} dt_{\omega(x, \gamma) \oplus \frac{\beta_1}{p}} \right]^q. \quad (4.15)$$

On the other hand, we have

$$x^{pq} = b_1^{pq} dt_{\frac{\beta_1}{pq}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^{pq}. \quad (4.16)$$

Therefore, the conclusion follows from (4.15), (4.16) and Lemma 29.

To prove 2, we can use the same method as before. Let $y = \sum_{k=1}^M e_k dt_{\varepsilon_k}$ be the decomposition of y , then

$$(xy)^p = \left(\sum_{j=1}^N \sum_{k=1}^M b_j e_k dt_{\beta_j \oplus \varepsilon_k} \right)^p.$$

Of course, $\beta_1 \oplus \varepsilon_1 > \beta_j \oplus \varepsilon_k$ for any j and k , and we can also consider

$$I := \{(j, k) \mid j = 1, \dots, N, \ k = 1, \dots, M, \ \beta_j \oplus \varepsilon_k \geq 1\},$$

so that to the sum

$$(xy)^p = \left(\sum_{(j,k) \in I} b_j e_k dt_{\beta_j \oplus \varepsilon_k} \right)^p$$

we can apply Lemma 32. We obtain

$$(xy)^p = b_1^p e_1^p dt_{\frac{\beta_1 \oplus \varepsilon_1}{p}} \cdot \left(1 + \sum_{\substack{(j,k) \in I \\ (j,k) \neq (1,1)}} \frac{b_j \cdot e_k}{b_1 \cdot e_1} dt_{\beta_j \oplus \varepsilon_k \ominus (\beta_1 \oplus \varepsilon_1)} \right)^p.$$

On the other hand, we have

$$\begin{aligned} x^p \cdot y^p &= b_1^p dt_{\frac{\beta_1}{p}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^p \\ &\quad \cdot e_1^p dt_{\frac{\varepsilon_1}{p}} \cdot \left(1 + \sum_{k=2}^M \frac{e_k}{e_1} dt_{\varepsilon_k \ominus \varepsilon_1} \right)^p = b_1^p e_1^p dt_{\frac{\beta_1 \oplus \varepsilon_1}{p}} \cdot (1+h)^p, \end{aligned}$$

where $h \in D_\infty$ is obtained from the product of the previous p -th powers with invertible bases. Once again, the conclusion follows from Lemma 29.

The proofs of 3 and 4 are straightforward, taking into account 1 and 2 so that, e.g., in the difference $(x^p)^q - x^{pq}$ there appear only infinitesimals of order greater than α_1 .

Finally, to prove 5, we only have to apply our Definition 25 of power:

$$\begin{aligned} x^p \cdot x^q &= b_1^p dt_{\frac{\beta_1}{p}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^p \\ &\quad \cdot b_1^q dt_{\frac{\beta_1}{q}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^q = \\ &= b_1^{p+q} dt_{\frac{\beta_1}{p+q}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^{p+q} = \\ &= x^{p+q}, \end{aligned}$$

which is the conclusion. \square

Remark 33. For generic $x, y \in D_\infty$, properties 3 and 4 of Theorem 31 cannot be improved. Indeed, if we had always

$$(x^p)^q = x^{pq} \quad \forall x \in D_\infty \quad \forall p, q > 0$$

we would have, as a consequence, the general validity of

$$\left(x^{\frac{1}{p}}\right)^p = x^{\frac{1}{p}p} = x \quad \forall x \in D_\infty \quad \forall p \in (0, 1]_{\mathbb{R}},$$

but we know that this property is not generally true.

Analogously, from the general validity of

$$(xy)^p = x^p \cdot y^p \quad \forall x \in D_\infty \quad \forall p > 0,$$

we would have

$$(x^2)^{\frac{1}{2}} = (x \cdot x)^{\frac{1}{2}} = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = (\sqrt{x})^2 = x \quad \forall x \in D_\infty,$$

but we know that this is not generally true.

Therefore, on the one hand these seem the best results attainable. However, it does not seem desirable to work with the equality $=_k$ up to infinitesimals of some order, in particular for such basic operations.

We have already noted that several counterexamples are of the form

$$\left[(dt_2 + dt)^2\right]^{\frac{1}{2}} = dt_2,$$

where we have terms like dt which are, intuitively, unnecessary with respect to the square. Our next aim is to formalize the idea of *incomplete term with respect to $(-)^{\frac{1}{p}}$* , and to prove that the usual properties of the powers hold, with the usual equality, if we use only Fermat reals without incomplete terms. We will also see why the name *incomplete term* seems a better choice than *unnecessary term*.

4.3 The notion of incomplete term

Let us start from the usual notations and hypotheses: $x = \sum_{j=1}^N b_j dt_{\beta_j}$ is the decomposition of x , and $p \in (0, 1]_{\mathbb{R}}$. In this decomposition, let us consider a term $dt_{\beta_{r+1}}$, for $r = 1, \dots, N-1$. The power $(-)^{1/p}$ is smooth, because $\frac{1}{p} > 1$, and, with the usual calculations, we can write

$$x^{\frac{1}{p}} = b_1^{\frac{1}{p}} dt_{p\beta_1} + \sum_{n=1}^k \binom{\frac{1}{p}}{n} \sum_{\substack{\gamma \in \mathbb{N}^{N-1} \\ |\gamma|=n}} \frac{n!}{\gamma!} \cdot b_1^{\frac{1}{p}-n} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_N^{\gamma_{N-1}} dt_{\omega(x, \gamma) \oplus p\beta_1}, \quad (4.17)$$

where

$$k := k_{x, \frac{1}{p}} := \begin{cases} \ulcorner \beta_2 \ominus \beta_1 \urcorner & \text{if } \frac{1}{p} \notin \mathbb{N} \\ \min \left(\ulcorner \beta_2 \ominus \beta_1 \urcorner, \frac{1}{p} \right) & \text{if } \frac{1}{p} \in \mathbb{N} \end{cases}$$

$$\omega(x, \gamma) := \bigoplus_{j=2}^N \frac{\beta_j \ominus \beta_1}{\gamma_{j-1}}.$$

We have two possibilities to identify the terms, like dt in $(dt_2 + dt)^2$, that are *unnecessary*, or, better, *incomplete*.

The first one is to say that a term of the type $dt_{\beta_{r+1}}$ *gives no contribution* whenever expanding the power $x^{1/p}$, it gives *always* zero summands, exactly like dt in $(dt_2 + dt)^2 = dt + (dt)^2 + 2 \cdot dt_2 \cdot dt = dt$. Putting it in negative form: if, expanding the power $x^{1/p}$, we have that *at least one summand*, obtained from $dt_{\beta_{r+1}}$, is not zero, then the term $dt_{\beta_{r+1}}$ *gives some contribution*, i.e. it is necessary.

A substantial objection against this idea, however, is the following: let us suppose that $dt_{\beta_{r+1}}$ is necessary, i.e. it gives some contribution. Then, the situation described above also includes the possibility that, in the expansion of the power $x^{1/p}$, the term $dt_{\beta_{r+1}}$ gives, e.g., only one contribution, whereas all the other terms involving $dt_{\beta_{r+1}}$ give zero. One of our first aims will be to prove that, if in the decomposition of x every term gives a contribution, then $(x^{1/p})^p = x$. The situation can actually be problematic because, following the previous extreme example, in $x^{1/p}$ “all the information concerning $dt_{\beta_{r+1}}$ ” is contained in the unique non zero term. For example,

$$x = dt_3 + dt_{\frac{3}{2}} \quad , \quad dt_{\beta_{r+1}} = dt_{\frac{3}{2}}$$

$$x^2 = dt_{\frac{3}{2}} + dt_{\frac{3}{4}} + 2 dt_{3 \oplus \frac{3}{2}} = dt_{\frac{3}{2}} + 2 dt.$$

Can the inverse operation $x^2 \mapsto \sqrt{x^2}$ reconstruct the whole initial information, about x , starting only from the unique non zero term $2 dt$ generated by $dt_{\beta_{r+1}}$?

$$\begin{aligned} \sqrt{x^2} &= dt_3 \cdot \left(1 + 2 dt_{1 \ominus \frac{3}{2}} \right)^{\frac{1}{2}} = \\ &= dt_3 \cdot \left(1 + \sum_{n=1}^{+\infty} \binom{\frac{1}{2}}{n} \cdot 2^n dt_{\frac{3}{n}} \right) = \\ &= dt_3 \cdot \left(1 + \frac{1}{2} 2 dt_3 - \frac{1}{8} 4 dt_{\frac{3}{2}} + \frac{1}{16} 8 dt \right) = \\ &= dt_3 + dt_{\frac{3}{2}} - \frac{1}{2} dt. \end{aligned}$$

This counterexample hence gives a negative answer to our question.

The second possibility of defining a precise notion of incomplete term arises from trying to prove the property $(x^{1/p})^p = x$ starting from a definition based on

the previous erroneous idea. We will say, intuitively, that $dt_{\beta_{r+1}}$ is incomplete whenever expanding the power $x^{1/p}$ gives *at least one* zero summand. For this reason, the term “incomplete” is better than “unnecessary”. Putting it in negative form: if expanding the power $x^{1/p}$, we have that *every summand*, obtained from $dt_{\beta_{r+1}}$, is not zero, then the term $dt_{\beta_{r+1}}$ *gives every contribution*, i.e. it is *complete*.

The particular situation of the leading term dt_{β_1} is more natural, and is tied to the idea that $(dt_{\beta_1})^{\frac{1}{p}} = dt_{p\beta}$, so that $(dt_{\beta_1})^{\frac{1}{p}} = 0$ if and only if $\beta_1 < \frac{1}{p}$.

All this motivates the following

Definition 34. Under the hypotheses introduced at the beginning of this section, we say that dt_{β_r} *loses information in $x^{\frac{1}{p}}$* , or that dt_{β_r} *is incomplete with respect to $x^{\frac{1}{p}}$* , if and only if the following conditions hold:

1. $r = 1 \implies \beta_1 < \frac{1}{p}$
2. If $r > 1$, then

$$\exists n = 1, \dots, k_{x, \frac{1}{p}} \exists \gamma \in \mathbb{N}^{N-1} : |\gamma| = n, \gamma_{r-1} \neq 0, \omega(x, \gamma) \oplus p\beta_1 < 1.$$

Let us analyze the condition 2 to see that it corresponds to our intuition:

- ‘ $\exists n = 1, \dots, k_{x, \frac{1}{p}} \exists \gamma \in \mathbb{N}^{N-1} : |\gamma| = n$ ’: looking at (4.17), we can say that this part of 2 corresponds to “at least one summand in the expansion of $x^{1/p}$ ”
- ‘ $\gamma_{r-1} \neq 0$ ’: “where the term dt_{β_r} appears”
- ‘ $\omega(x, \gamma) \oplus p\beta_1 < 1$ ’: “is zero”.

Consider the case $p = \frac{1}{q}$, where $q \in \mathbb{N}_{>0}$. In this case, the power $x^{1/p}$ becomes x^q and hence we obtain an equivalent formulation starting from the multinomial formula:

$$x^q = \left(\sum_{j=1}^N b_j dt_{\beta_j} \right)^q = \sum_{\substack{\eta \in \mathbb{N}^N \\ |\eta| = q}} \frac{q!}{\eta!} b_1^{\eta_1} \dots b_N^{\eta_N} dt_{\bigoplus_{j=1}^N \frac{\beta_j}{\eta_j}}.$$

In fact, we have

Theorem 35. If $p = \frac{1}{q}$, $q \in \mathbb{N}_{>0}$, with $q \leq \lceil \beta_2 \ominus \beta_1 \rceil$, then we have that

$$dt_{\beta_{r+1}} \text{ loses information in } x^q$$

if and only if

$$\exists \eta \in \mathbb{N}^N : |\eta| = q, \eta_{r+1} \neq 0, \bigoplus_{j=1}^N \frac{\beta_j}{\eta_j} < 1. \quad (4.18)$$

Proof: We first compute the term $\omega(x, \gamma) \oplus \frac{\beta_1}{q}$ of Definition 34, in the case $|\gamma| = q - a$, where $a \in \mathbb{N}_{\leq q}$:

$$\begin{aligned}
\left(\omega(x, \gamma) \oplus \frac{\beta_1}{q} \right)^{-1} &= \left(\frac{\beta_1}{q} \oplus \bigoplus_{j=2}^N \frac{\beta_j \ominus \beta_1}{\gamma_{j-1}} \right)^{-1} = \\
&= \frac{q}{\beta_1} + \sum_{j=2}^N \frac{\gamma_{j-1}}{\beta_j \ominus \beta_1} = \\
&= \frac{q}{\beta_1} + \sum_{j=2}^N \gamma_{j-1} \cdot \left(\frac{1}{\beta_j} - \frac{1}{\beta_1} \right) = \\
&= \frac{q}{\beta_1} + \sum_{j=2}^N \frac{\gamma_{j-1}}{\beta_j} - \frac{q}{\beta_1} + \frac{a}{\beta_1},
\end{aligned}$$

where the last equality is due to the hypothesis $|\gamma| = q - a$. Therefore

$$\omega(x, \gamma) \oplus \frac{\beta_1}{q} = \frac{\beta_1}{a} \oplus \bigoplus_{j=2}^N \frac{\beta_j}{\gamma_{j-1}} \quad \text{if } |\gamma| = q - a. \quad (4.19)$$

To prove that (4.18) is necessary, we start from the hypothesis that there exist $n = 1, \dots, k_{x, \frac{1}{p}} = \min(\ulcorner \beta_2 \ominus \beta_1 \urcorner, q) = q$ and there exists $\gamma \in \mathbb{N}^{N-1}$ such that

$$|\gamma| = n \quad , \quad \gamma_r \neq 0 \quad , \quad \omega(x, \gamma) \oplus \frac{\beta_1}{q} < 1.$$

Set

$$\eta_j := \begin{cases} q - n & \text{if } j = 1 \\ \gamma_{j-1} & \text{if } j = 2, \dots, N \end{cases}$$

Then $\eta \in \mathbb{N}^N$, $|\eta| = |\gamma| + q - n = q$, and $\eta_{r+1} = \gamma_r \neq 0$. Finally, from (4.19), with $a := q - n$, we obtain

$$\bigoplus_{j=1}^N \frac{\beta_j}{\eta_j} = \frac{\beta_1}{q - n} \oplus \bigoplus_{j=2}^N \frac{\beta_j}{\gamma_{j-1}} = \omega(x, \gamma) \oplus \frac{\beta_1}{q} < 1,$$

which concludes the first part of our proof.

To prove that (4.18) is sufficient to obtain that $dt_{\beta_{r+1}}$ is incomplete, we consider η as in (4.18) and set $\gamma := (\eta_2, \dots, \eta_N) \in \mathbb{N}^{N-1}$. Then

$$|\gamma| = |\eta| - \eta_1 = q - \eta_1 \leq q = \min(\ulcorner \beta_2 \ominus \beta_1 \urcorner, q) = k_{x, \frac{1}{p}}$$

and $\gamma_r = \eta_{r+1} \neq 0$. Using (4.19) with $a := \eta_1 \leq |\eta| = q$, we have

$$\omega(x, \gamma) \oplus \frac{\beta_1}{q} = \frac{\beta_1}{\eta_1} \oplus \bigoplus_{j=2}^N \frac{\beta_j}{\gamma_{j-1}} = \bigoplus_{j=1}^N \frac{\beta_j}{\eta_j} < 1,$$

which concludes our proof. \square

The following theorem can be viewed as a validation of our definition of incomplete term.

Theorem 36. *Let $x \in D_\infty$ be a strictly positive infinitesimal, and let $p \in \mathbb{R}_{>0}$. Suppose that in the decomposition of x no term loses information in $x^{\frac{1}{p}}$, then*

$$\left(x^{\frac{1}{p}}\right)^p = x.$$

Therefore, the power $(-)^p$ is an injection on the set

$$\left\{x \in D_\infty \mid \text{all terms of } x \text{ are } (-)^{\frac{1}{p}} \text{ complete}\right\} \cup \{y \in \bullet\mathbb{R} \mid {}^\circ y \neq 0\}.$$

Proof: If $p > 1$, setting $q := \frac{1}{p} < 1$, the conclusion follows from Theorem 27:

$$(x^q)^{\frac{1}{q}} = x = \left(x^{\frac{1}{p}}\right)^p. \text{ Therefore, the only interesting case is } p < 1.$$

If, in the decomposition of x , we have only $N = 1$ term, then, by hypothesis, this term is not incomplete. Taking the negation of Definition 34 for $r = 1$, we obtain $\beta_1 \geq \frac{1}{p}$ and

$$\left(x^{\frac{1}{p}}\right)^p = \left[(b_1 dt_{\beta_1})^{\frac{1}{p}}\right]^p = \left(b_1^{\frac{1}{p}} dt_{p\beta_1}\right)^p = b_1 dt_{\beta_1} = x.$$

Let us observe that since $p\beta_1 \geq 1$, the expression $b_1^{\frac{1}{p}} dt_{p\beta_1}$ is a decomposition, so that in taking its p -th power, we have applied Definition 25. We can hence suppose $N > 1$.

As usual, we refer to (4.17). By hypothesis, every term dt_{β_j} is complete, which means

$$p\beta_1 \geq 1$$

and, for every $j = 2, \dots, N$, $n = 1, \dots, k$, and $\gamma \in \mathbb{N}^{N-1}$ we must have

$$|\gamma| = n, \gamma_{j-1} \neq 0 \implies \omega(x, \gamma) \oplus p\beta_1 \geq 1.$$

This implies that in (4.17) the greatest infinitesimal is $dt_{p\beta_1}$ and every summand is not zero. Therefore, to compute $\left(x^{\frac{1}{p}}\right)^p$, we can use Lemma 32:

$$\begin{aligned} \left(x^{\frac{1}{p}}\right)^p &= b_1 dt_{\beta_1} \cdot \left(1 + \sum_{n=1}^k \binom{\frac{1}{p}}{n} \sum_{|\gamma|=n} \frac{n!}{\gamma!} \cdot b_1^{-n} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_N^{\gamma_{N-1}} dt_{\omega(x, \gamma)}\right)^p \\ &=: b_1 dt_{\beta_1} \cdot (1 + h)^p. \end{aligned}$$

We have to prove that

$$\begin{aligned} b_1 dt_{\beta_1} \cdot (1 + h)^p &= x \\ &= b_1 dt_{\beta_1} + b_2 dt_{\beta_2} + \dots + b_N dt_{\beta_N} \\ &=: b_1 dt_{\beta_1} + k. \end{aligned}$$

To this end, we use the following

Lemma 37. Let $h, k \in D_\infty$ and $p, b, \beta \in \mathbb{R}_{\neq 0}$ such that $p < 1$ and $\beta > \omega(k)$. Define

$$\frac{k}{b \, dt_\beta} := \sum_{i=1}^N \frac{a_i}{b} \, dt_{\alpha_i \ominus \beta},$$

where $k = \sum_{i=1}^N a_i \, dt_{\alpha_i}$ is the decomposition of k . Then

$$1 + h = \left(1 + \frac{k}{b \, dt_\beta}\right)^{\frac{1}{p}} \implies b \, dt_\beta \cdot (1 + h)^p = b \, dt_\beta + k. \quad (4.20)$$

Therefore, to obtain our conclusion it suffices to verify the assumption of (4.20), that is

$$\begin{aligned} 1 + \sum_{n=1}^k \binom{\frac{1}{p}}{n} \sum_{|\gamma|=n} \frac{n!}{\gamma!} \cdot b_1^{-n} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_N^{\gamma_{N-1}} \, dt_{\omega(x, \gamma)} &= \\ &= \left(1 + \frac{b_2}{b_1} \, dt_{\beta_2 \ominus \beta_1} + \dots + \frac{b_N}{b_1} \, dt_{\beta_N \ominus \beta_1}\right)^{\frac{1}{p}}. \end{aligned}$$

Indeed,

$$\begin{aligned} \left(1 + \frac{b_2}{b_1} \, dt_{\beta_2 \ominus \beta_1} + \dots + \frac{b_N}{b_1} \, dt_{\beta_N \ominus \beta_1}\right)^{\frac{1}{p}} &= 1 + \sum_{n=1}^k \binom{\frac{1}{p}}{n} \left(\sum_{j=2}^N \frac{b_j}{b_1} \, dt_{\beta_j \ominus \beta_1}\right)^n \\ &= 1 + \sum_{n=1}^k \binom{\frac{1}{p}}{n} \sum_{|\gamma|=n} \frac{n!}{\gamma!} \cdot \left(\frac{b_2}{b_1}\right)^{\gamma_1} \cdot \dots \cdot \left(\frac{b_N}{b_1}\right)^{\gamma_{N-1}} \, dt_{\bigoplus_{j=2}^N \frac{\beta_j \ominus \beta_1}{\gamma_{j-1}}} \\ &= 1 + \sum_{n=1}^k \binom{\frac{1}{p}}{n} \sum_{|\gamma|=n} \frac{n!}{\gamma!} \cdot b_1^{-n} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_N^{\gamma_{N-1}} \, dt_{\omega(x, \gamma)} = 1 + h, \end{aligned}$$

as claimed. \square

Proof of Lemma 37: It suffices to note that the powers $(-)^p$ and $(-)^{\frac{1}{p}}$ are smooth if applied to invertible Fermat reals. By the elementary transfer theorem, they hence have all the usual properties, so that we can write

$$(1 + h)^p = \left[\left(1 + \frac{k}{b \, dt_\beta}\right)^{\frac{1}{p}}\right]^p = 1 + \frac{k}{b \, dt_\beta}.$$

Thus,

$$\begin{aligned} b \, dt_\beta \cdot (1 + h)^p &= b \, dt_\beta + b \, dt_\beta \cdot \frac{k}{b \, dt_\beta} = \\ &= b \, dt_\beta + b \, dt_\beta \cdot \left(\sum_{i=1}^N \frac{a_i}{b} \, dt_{\alpha_i \ominus \beta}\right) = \\ &= b \, dt_\beta + k \end{aligned}$$

\square

Remark 38. To simplify the notations, let us define

$$x \text{ is } (-)^{\frac{1}{p}}\text{-complete} \quad :\Longleftrightarrow \quad \forall j = 1, \dots, N : \text{ dt}_{\beta_j} \text{ is complete w.r.t. } x^{\frac{1}{p}}$$

$$\text{Compl}\left(\frac{1}{p}\right) := \left\{ x \in D_\infty \mid x \text{ is } (-)^{\frac{1}{p}}\text{-complete} \right\} \cup \{ y \in \bullet\mathbb{R} \mid {}^\circ y \neq 0 \}.$$

Then the map

$$(-)^p|_{\text{Compl}(\frac{1}{p})} : \text{Compl}\left(\frac{1}{p}\right) \longrightarrow \bullet\mathbb{R}$$

is injective. Moreover, the map

$$(-)^p : \bullet\mathbb{R} \longrightarrow \bullet\mathbb{R}$$

is surjective. However, the power $(-)^p$ doesn't map $\text{Compl}\left(\frac{1}{p}\right)$ onto itself. In fact, if $p = \frac{1}{2}$ and $y = \text{dt}_2$, then $y^{\frac{1}{p}} = y^2 = \text{dt}$ so that every summand in $y^{\frac{1}{p}}$ is nonzero and hence $y \in \text{Compl}\left(\frac{1}{p}\right)$. This y is therefore in the codomain, but it is not of the form $y = x^p$ for $x \in \text{Compl}\left(\frac{1}{p}\right)$, as otherwise $y^2 = \left(x^{\frac{1}{2}}\right)^2 = x$, so that $x = \text{dt}$ and we would also have $\text{dt} \in \text{Compl}(2)$, contradicting $(\text{dt})^2 = 0$.

We can now prove that complete Fermat infinitesimals have very favorable properties related to powers.

Theorem 39. *Let $x \in D_\infty$ be a strictly positive infinitesimal, and $p, q \in \mathbb{R}_{>0}$. Then*

$$x \text{ is } (-)^p\text{-complete} \quad \implies \quad (x^p)^q = x^{pq}.$$

Proof: Let us start from the usual formula (4.11) applied to x^p :

$$x^p = b_1^p \text{dt}_{\frac{\beta_1}{p}} + \sum_{n=1}^{k_{x,p}} \binom{p}{n} \sum_{\gamma \in \Gamma_{x,n}} \frac{n!}{\gamma!} \cdot b_1^{p-n} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_N^{\gamma_{N-1}} \text{dt}_{\omega(x,\gamma) \oplus \frac{\beta_1}{p}}.$$

By hypothesis, x is $(-)^{\frac{1}{p}}$ -complete, that is $\frac{\beta_1}{p} \geq 1$, and for every $j = 2, \dots, N$, $n = 1, \dots, k_{x,p}$, and $\gamma \in \mathbb{N}^{N-1}$ we have

$$|\gamma| = n, \gamma_{j-1} \neq 0 \quad \implies \quad \omega(x, \gamma) \oplus \frac{\beta_1}{p} \geq 1. \quad (4.21)$$

This property implies that $\Gamma_{x,n}$ is trivial, in fact, in general

$$\forall a, b \in \mathbb{R} : a < 1 \quad \implies \quad (a \oplus b)^{-1} = \frac{1}{a} + \frac{1}{b} > \frac{1}{a} > 1.$$

Therefore

$$\forall n = 1, \dots, k_{x,p} \quad \forall \gamma \in \mathbb{N}^{N-1} : |\gamma| = n \quad \implies \quad \omega(x, \gamma) \geq 1,$$

$$x^p = b_1^p dt_{\frac{\beta_1}{p}} + \sum_{n=1}^{k_{x,p}} \binom{p}{n} \cdot \sum_{\substack{\gamma \in \mathbb{N}^{N-1} \\ |\gamma|=n}} \frac{n!}{\gamma!} \cdot b_1^{p-n} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_N^{\gamma_{N-1}} dt_{\omega(x,\gamma) \oplus \frac{\beta_1}{p}}. \quad (4.22)$$

By (4.21), every summand in (4.22) has order greater or equal to 1. We can hence apply Lemma 32 obtaining

$$\begin{aligned} (x^p)^q &= b_1^{pq} dt_{\frac{\beta_1}{pq}} \cdot \left(1 + \sum_{n=1}^{k_{x,p}} \binom{p}{n} \sum_{\substack{\gamma \in \mathbb{N}^{N-1} \\ |\gamma|=n}} \frac{n!}{\gamma!} \cdot b_1^{-n} \cdot b_2^{\gamma_1} \cdot \dots \cdot b_N^{\gamma_{N-1}} dt_{\omega(x,\gamma)} \right)^q \\ &= b_1^{pq} dt_{\frac{\beta_1}{pq}} \cdot \left[\left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^p \right]^q \\ &= b_1^{pq} dt_{\frac{\beta_1}{pq}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^{pq} \\ &= x^{pq}. \end{aligned}$$

Here the property $[(1+h)^p]^q = (1+h)^{pq}$, for $h \in D_\infty$, holds because the base is invertible. \square

The following result does not depend on the notion of complete term, but supposes that in the product $x \cdot y$ no term becomes zero.

Theorem 40. *Let $x, y \in \bullet\mathbb{R}_{>0}$ and $p \in \mathbb{R}_{>0}$, such that*

$$\forall i = 1, \dots, N_y \quad \forall j = 1, \dots, N_x : \omega_i(y) \oplus \omega_j(x) \geq 1, \quad (4.23)$$

where N_x and N_y are the number of summands in the decompositions of x, y respectively. Then

$$(x \cdot y)^p = x^p \cdot y^p.$$

Let us observe that the hypothesis (4.23) reduces to the usual $(-)^2$ -completeness in case $x = y$, i.e. in case of the property $(x^2)^p = x^p \cdot x^p$.

Proof: We will proceed for $x, y \in D_\infty$, the proof being analogous if x or y is invertible. Let $x = \sum_{j=1}^N b_j dt_{\beta_j}$ and $y = \sum_{i=1}^M a_i dt_{\alpha_i}$ be the decompositions of x and y . Then

$$x \cdot y = \sum_{i,j} b_j a_i dt_{\alpha_i \oplus \beta_j}. \quad (4.24)$$

By hypothesis, every summand in this sum is nonzero. Of course, $\alpha_1 \oplus \beta_1$ is the leading term in (4.24), and we can hence apply Lemma 32 to obtain

$$(xy)^p = a_1^p b_1^p dt_{\frac{\alpha_1 \oplus \beta_1}{p}} \cdot \left(1 + \sum_{\substack{i,j \\ (i,j) \neq (1,1)}} \frac{b_j a_i}{b_1 a_1} dt_{\alpha_i \oplus \beta_j \ominus (\alpha_1 \oplus \beta_1)} \right)^p. \quad (4.25)$$

On the other hand

$$x^p = b_1^p dt_{\frac{\beta_1}{p}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^p \quad (4.26)$$

$$y^p = a_1^p dt_{\frac{\alpha_1}{p}} \cdot \left(1 + \sum_{i=2}^M \frac{a_i}{a_1} dt_{\alpha_i \ominus \alpha_1} \right)^p. \quad (4.27)$$

The equality (4.25) can also be written as

$$(xy)^p = a_1^p b_1^p dt_{\frac{\alpha_1 \oplus \beta_1}{p}} \cdot \left(1 + \sum_j \frac{b_j a_1}{b_1 a_1} dt_{\alpha_1 \oplus \beta_j \ominus (\alpha_1 \oplus \beta_1)} + \sum_i \frac{b_1 a_i}{b_1 a_1} dt_{\alpha_i \oplus \beta_1 \ominus (\alpha_1 \oplus \beta_1)} + \sum_{\substack{i \geq 2 \\ j \geq 2}} \frac{b_j a_i}{b_1 a_1} dt_{\alpha_i \oplus \beta_j \ominus (\alpha_1 \oplus \beta_1)} \right)^p,$$

which is the same result obtained from multiplying (4.26) and (4.27). \square

4.4 Roots are not smooth

Here we prove that the power function

$$(-)^p : \bullet\mathbb{R}_{>0} \longrightarrow \bullet\mathbb{R} \quad , \quad p < 1$$

is not (non standard) smooth on $D_\infty \setminus \{0\}$ (for the notion of non standard smoothness, see [10, 9]). This is a naturally expected result, because the corresponding derivative should be

$$\frac{dx^p}{dx}(h) = p \cdot h^{p-1} = \frac{p}{h^{1-p}},$$

and, intuitively, $\frac{1}{h^{1-p}}$ is an infinite, whereas in $\bullet\mathbb{R}$ we obviously do not have infinities. Therefore, the theory of Fermat reals should be sufficiently complete to prove that the power function $(-)^p$ is not smooth at $h \in D_\infty \setminus \{0\}$. Indeed,

from the generalized Taylor formula (see Theorem 12.1.3 and Definition 12.2.7 in [9]), if we assume that $(-)^p$ is smooth, we would have

$$\exists m \in {}^\bullet\mathbb{R}^\mathbb{N} : \forall k \in D_\infty : (h+k)^p = \sum_{j=0}^{+\infty} \frac{h^j}{j!} \cdot m_j. \quad (4.28)$$

For the sake of completeness, we recall that this sequence $m = (m_j)_{j \in \mathbb{N}}$ is unique up to first order infinitesimals, i.e. if $\bar{m} = (\bar{m}_j)_{j \in \mathbb{N}}$ verifies (4.28), then $m_j =_1 \bar{m}_j$ for every $j \in \mathbb{N}$.

Set $s := -\text{sgn}(h) \in \{+1, -1\}$, and take $k = s \, dt$ in (4.28). We have that $h + s \, dt \neq 0$ because $\text{sgn}(h) \neq \text{sgn}(s \, dt)$, and taking into account that $m_0 = 0^p = 0$, we obtain

$$(h + s \, dt)^p = m_0 + s \, dt \cdot m_1 = s m_1 \, dt \in I_1 = \{x \in {}^\bullet\mathbb{R} \mid \omega(x) \leq 1\}.$$

Now, the order of $(h + s \, dt)^p$ is $\frac{\max(\omega(h), 1)}{p} = \frac{\omega(h)}{p}$ because $h + s \, dt \neq 0$ and since always $\omega(h) \geq 1$. Therefore, we would have $\frac{\omega(h)}{p} \leq 1$, i.e. $p \geq \omega(h) \geq 1$, whereas we have supposed $p < 1$.

4.5 An application to the infinitesimal Taylor formula with fractional derivatives

Using powers h^p of infinitesimals $h \in D_\infty$, we can prove an infinitesimal Taylor formula with fractional derivatives in a straightforward manner. This further underlines the ease of translating classical results using the infinitesimal language of the ring of Fermat reals. Frequently, these translations are really faithful to the informal use sometimes appearing in applications. Let us note that the same translations are not so easily performed in algebraic models of infinitesimals, like in Synthetic Differential Geometry (see, e.g., [21], [15]) or in Levi-Civita fields ([26, 25]) or Weil functors ([16, 18]).

We start with some definitions and a theorem, taken from [23].

Definition 41. If $\alpha \in \mathbb{R}$, we will denote with $\mathcal{C}_\alpha(\mathbb{R}_{>0}, \mathbb{R})$ the set of all the functions $f : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ that can be written as

$$f(x) = x^p \cdot f_1(x) \quad \forall x \in \mathbb{R}_{>0},$$

for some $p > \alpha$ and some continuous function $f_1 \in \mathcal{C}^0(\mathbb{R}_{>0}, \mathbb{R})$. Moreover, for every $m \in \mathbb{N}_{>0} \cup \{\infty\}$ we also set

$$\mathcal{C}_\alpha^m(\mathbb{R}_{>0}, \mathbb{R}) := \left\{ f \in \mathbb{R}_{>0} \rightarrow \mathbb{R} \mid f^{(m)} \in \mathcal{C}_\alpha(\mathbb{R}_{>0}, \mathbb{R}) \right\}.$$

Secondly, we define the Riemann-Liouville integral operator of order $\alpha > 0$ with $a \geq 0$.

Definition 42. Let $\alpha \in \mathbb{R}_{>0}$, $a \in \mathbb{R}_{\geq 0}$ and $f \in \mathcal{C}_\alpha(\mathbb{R}_{>0}, \mathbb{R})$, then

$$\begin{aligned}
J_a^0 f(x) &:= f(x) \\
J_a^\alpha f(x) &:= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \quad \forall x \in \mathbb{R}_{>a}.
\end{aligned} \tag{4.29}$$

Here Γ denotes the gamma function. To derive the fractional Taylor formula, we need the Caputo fractional derivative.

Definition 43. Let $\alpha \in \mathbb{R}_{>0}$, $a \in \mathbb{R}_{\geq 0}$, and $f \in \mathcal{C}_{-1}^m(\mathbb{R}_{>0}, \mathbb{R})$. For simplicity of notations, let $m := \lceil \alpha \rceil$ be the ceiling of α . Then

$$D_a^\alpha f : \mathbb{R}_{\geq a} \longrightarrow \mathbb{R} \tag{4.30}$$

$$D_a^\alpha f(x) := J_a^{m-\alpha} f^{(m)}(x) \quad \forall x \geq a. \tag{4.31}$$

Finally, we set

$$D_a^{n,\alpha} = D_a^\alpha \circ \dots \circ D_a^\alpha \quad \forall n \in \mathbb{N}_{>0}.$$

From (4.30) and (4.29) we therefore have

$$D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

where $m = \lceil \alpha \rceil$.

The non-infinitesimal version of the generalized Taylor formula with fractional derivatives is the following. For its proof, see [23].

Theorem 44. Let $\alpha, a, b \in \mathbb{R}$, and $n \in \mathbb{N}$, with $0 \leq a < b$ and $0 < \alpha \leq 1$. Consider a continuous function $f \in \mathcal{C}_0([a, b], \mathbb{R})$ such that

$$D_a^{k,\alpha} f \in \mathcal{C}_0([a, b], \mathbb{R}) \quad \forall k = 0, \dots, n+1.$$

Then for every $x \in (a, b]$ there exists $\xi \in [a, x]$ such that

$$f(x) = \sum_{i=0}^n \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} D_a^{i,\alpha} f(a) + \frac{D_a^{n+1,\alpha} f(\xi)}{\Gamma((n+1)\alpha+1)} \cdot (x-a)^{(n+1)\alpha}.$$

In our framework we are able to prove a corresponding infinitesimal Taylor formula for the following class of smooth functions:

Definition 45. In the hypothesis of the previous theorem, we set

$$\mathcal{C}_\alpha^\infty([a, b], \mathbb{R}) := \{f \in \mathcal{C}^\infty([a, b], \mathbb{R}) \mid D_a^{k,\alpha} f \in \mathcal{C}_0([a, b], \mathbb{R}) \quad \forall k \in \mathbb{N}\}.$$

Finally, we can state the main result of this section:

Theorem 46. Let $\alpha, a', a, b \in \mathbb{R}$, and $n \in \mathbb{N}$, with $0 \leq a' < a < b$ and $0 < \alpha \leq 1$. Consider a smooth function $f \in \mathcal{C}_\alpha^\infty([a, b], \mathbb{R})$, then

$$\forall h \in D_{(n+1)\alpha-1} : f(a+h) = \sum_{i=0}^n \frac{h^{i\alpha}}{\Gamma(i\alpha+1)} D_a^{i,\alpha} f(a).$$

Proof: Let $h = \sum_{j=1}^N b_j dt_{\beta_j}$ be the decomposition of the infinitesimal $h \in D_{(n+1)\alpha}$. By Definition 25, we have

$$h^{i\alpha} = b_1^{i\alpha} dt_{\frac{\beta_1}{i\alpha}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} dt_{\beta_j \ominus \beta_1} \right)^{i\alpha} \quad \forall i = 0, \dots, n.$$

This means, using an innocuous abuse of language, that $h_t := \sum_{j=1}^N b_j t^{\frac{1}{\beta_j}}$ and

$$h_t^{i\alpha} := b_1^{i\alpha} t^{\frac{i\alpha}{\beta_1}} \cdot \left(1 + \sum_{j=2}^N \frac{b_j}{b_1} t^{\frac{1}{\beta_j \ominus \beta_1}} \right)^{i\alpha} \quad \forall t \geq 0$$

are little-oh polynomials representing the Fermat real h and $h^{i\alpha}$ respectively, and

$$(h_t)^{i\alpha} = h_t^{i\alpha} \quad \forall t \geq 0.$$

For $t \geq 0$ sufficiently small, we have $a' < a + h_t < b$, and we can apply Theorem 44 at the point a , obtaining

$$f(a + h_t) = \sum_{i=0}^n \frac{h_t^{i\alpha}}{\Gamma(i\alpha + 1)} D_a^{i,\alpha} f(x) + \frac{D_a^{n+1,\alpha} f(\xi_t)}{\Gamma((n+1)\alpha + 1)} \cdot h_t^{(n+1)\alpha} \quad \xi_t \in [a, a + h_t]. \quad (4.32)$$

Now, $h \in D_{(n+1)\alpha-1}$ so that $\omega(h) < (n+1)\alpha$ and $h^{(n+1)\alpha} = 0$, that is

$$\lim_{t \rightarrow 0^+} \frac{h_t^{(n+1)\alpha}}{t} = 0. \quad (4.33)$$

Considering that $D_a^{n+1,\alpha} f$ is continuous on $[a, b]$ and that $\xi_t \in [a, b]$, from (4.33) and (4.32) we obtain

$$f(a + h_t) = \sum_{i=0}^n \frac{h_t^{i\alpha}}{\Gamma(i\alpha + 1)} D_a^{i,\alpha} f(x) + o(t) \quad \text{as } t \rightarrow 0^+,$$

yielding the claim. \square

5 Computer implementation

The definition of the ring of Fermat reals is highly constructive. Therefore, using object oriented programming, it is not hard to write a computer code corresponding to $\bullet\mathbb{R}$. We realized a first version of this software using Matlab R2010b.

The constructor of a Fermat real is $\mathbf{x} = \text{FermatReal}(\mathbf{s}, \mathbf{w}, \mathbf{r})$, where \mathbf{s} is the $n+1$ double vector of standard parts ($\mathbf{s}(1)$ is the standard part ${}^\circ x$) and \mathbf{w} is the double vector of orders ($\mathbf{w}(1)$ is the order $\omega(x)$ if $x \in \bullet\mathbb{R} \setminus \mathbb{R}$, otherwise $\mathbf{w} = []$ is

the empty vector). The last input `r` is a logical variable and assumes the value `true` if we want that the display of the number `x` is realized using the Matlab `rats` function for both its standard parts and orders. In this way, the number will be displayed using continued fraction approximations and therefore, in many cases, the calculations will be exact. These inputs are the basic methods of every Fermat real, and can be accessed using the `subsref`, and `subsasgn`, notations `x.stdParts`, `x.orders`, `x.rats`. The function `w=orders(x)` gives exactly the double vector `x.orders` if $x \in \bullet\mathbb{R} \setminus \mathbb{R}$ and 0 otherwise.

The function `dt(a)`, where `a` is a double, constructs the Fermat real dt_a . Because we have overloaded all the algebraic operations, like `x+y`, `x*y`, `x-y`, `-x`, `x==y`, `x~=y`, `x<y`, `x<=y`, `x^y`, we can define a Fermat real e.g. using an expression of the form `x=2+3*dt(2)-1/3*dt(1)`, which corresponds to `x=FermatReal([2 3 -1/3],[2 1],true)`.

We have also realized the function `y=decomposition(x)`, which gives the decomposition of the Fermat real `x`, `abs(x)`, `log(x)`, `exp(x)`, `isreal(x)`, `is-infinitesimal(x)`, `isinvertible(x)`.

The logical function `v=eqUpTo(k,x,y)` corresponds to $x =_k y$.

The ratio `x/y` (see Theorem 22) has been implemented for `x` and `y` infinitesimals and `y~=0`, or in case `y` is invertible. Finally, the function `y=ext(f,x)`, corresponds to $\bullet f(x)$ and has been realized using the evaluation of the symbolic Taylor formula of the inline function `f`.

The functions `dF` and `dOmega` correspond, respectively, to the Fermat and the omega distance, while `x^p`, `sqrt(x)` and `nthroot(x,n)` have been realized both for `x` infinitesimal or invertible using the formulas we have derived in the present work.

Using these tools, we can easily find, e.g., that

$$\frac{\sin(\sqrt{dt_3 + 2dt_2})}{\cos(\sqrt[3]{-4dt})} = dt_6 + dt_3 - \frac{2}{3}dt_2 + \frac{1096}{2787}dt_{\frac{6}{5}} + \frac{1234}{913}dt.$$

This corresponds to the following Matlab code:

```
>> x=sqrt(dt(3)+2*dt(2))
x =
dt_6 + dt_3 - 1/2*dt_2 + 1/2*dt_3/2 - 5/8*dt_6/5
>> y=nthroot(-4*dt(1),3)
y =
-1008/635*dt_3
>> g=inline('cos(y)')
g =
Inline function: g(y) = cos(y)
>> f=inline('sin(x)')
f =
Inline function: f(x) = sin(x)
>> decomposition(ext(f,x)/ext(g,y))
ans =
dt_6 + dt_3 - 2/3*dt_2 + 1096/2787*dt_6/5 + 1234/913*dt
```

Up to now, this code has been written only to show concretely the possibilities of the ring $\bullet\mathbb{R}$. On the other hand, it is clear that it is possible to write it with a more specific aim. For example, as in case of the Levi-Civita field ([3, 25]) possible applications of a specifically rewritten code include automatic differentiation theory. Let us note that, even if the theory of Fermat reals applies to smooth functions, a full treatment of right and left sided derivatives is possible ([9]), so that the theory can be applied consistently also to piecewise smooth functions. Finally, the use of nilpotent elements permits to fully justify that every derivative estimation of a computer function ([25]) reduces to a finite number of algebraic calculations.

The Matlab source code is freely available under open-source licence, and can be requested from the authors of the present article.

6 Conclusions

Usually, it is common to study extended structures, like the ring of Fermat reals $\bullet\mathbb{R}$, using suitable extensions of well established notions. For example, it is more natural to search for metrics of the form $d : \bullet\mathbb{R} \times \bullet\mathbb{R} \longrightarrow \bullet\mathbb{R}$ than for standard metrics on the set $\bullet\mathbb{R}$. We have shown that it is possible, and also very natural, to define standard topological structures on the ring $\bullet\mathbb{R}$ having very favorable relationships with various aspects of differential calculus on $\bullet\mathbb{R}$. This allows a better dialog with mathematicians not already familiar with the theory of Fermat reals and underlines that this ring is not “non standard”.

Moreover, with the present work, we are continuing our program to define a meaningful and powerful ring with infinitesimals using only very well behaved representative functions for new numbers. If one thinks at non standard analysis or Colombeau’s ring of generalized numbers, it becomes clear that this quest is nontrivial. As a consequence, we have been able to characterize ideals of the ring $\bullet\mathbb{R}$ in a very simple and descriptive way.

Finally, we have proved that nilpotent elements and arbitrary roots can coexist very well, even if this seems impossible at a first glance. This is a very important step toward the idea of using nilpotent infinitesimals for stochastic calculus. For example, based on a very helpful discussion with N. Blagowest (Department of Physics, K. Preslawki University, Bulgaria) we may call *Ito process* any (deterministic) function $x : \bullet\mathbb{R} \longrightarrow \bullet\mathbb{R}$ such that

$$\forall t \in \mathbb{R} \ \forall h \in D_{\frac{1}{2}} : x(t+h) = x(t) + v[t, x(t)] \cdot h + \lambda\sqrt{h}.$$

In this approach, the deep mathematical problem is that there doesn’t exist a non trivial smooth function that verifies such a definition. Of course, we need continuous but nowhere differentiable functions and hence, we need to extend the ring $\bullet\mathbb{R}$ by suitable infinities. Indeed, this is indispensable if we want to consider the derivatives of the function x . Therefore, the new problem to face becomes: can infinities coexist as reciprocals of nilpotent infinitesimals? This question will be the subject of future work.

References

- [1] R. Abraham, J.E. Marsden, and T. Ratiu. *Manifolds, Tensors, Analysis and Applications*. Springer-Verlag, second edition, 1988.
- [2] W. Bertram. *Differential Geometry, Lie Groups and Symmetric Spaces over General Base Fields and Rings*. American Mathematical Society, Providence, 2008.
- [3] M. Berz, G. Hoffstatter, W. Wan, K. Shamseddine and K. Makino *COSY INFINITY and its Applications to Nonlinear Dynamics*. Chapter Computational Differentiation: Techniques, Applications, and Tools, pages 363–367. SIAM, Philadelphia, Penn, 1966.
- [4] A. Bigard, K. Keimel, S. Wolfenstein, *Groupes et anneaux réticulés*, Lecture Notes in Mathematics, Vol. 608, Springer-Verlag, Berlin, 1977.
- [5] J.F. Colombeau, *New generalized functions and multiplication of distributions*, North-Holland Mathematics Studies, **84**. North-Holland, Amsterdam, 1984.
- [6] J.F. Colombeau. *Multiplication of Distributions*. Springer, Berlin, 1992.
- [7] J.H. Conway. *On Numbers and Games*. Number 6 in L.M.S. monographs. Academic Press, London & New York, 1976.
- [8] P. Ehrlich. An alternative construction of Conway’s ordered field No. *Algebra Universalis*, 25:7–16, 1988.
- [9] P. Giordano. Fermat reals: Nilpotent infinitesimals and infinite dimensional spaces. arXiv:0907.1872, July 2009.
- [10] P. Giordano. Fermat-Reyes method in the ring of Fermat reals. Submitted to *Advances in Mathematics*, 2010.
- [11] P. Giordano. Infinitesimals without logic. *Russian Journal of Mathematical Physics*, 17(2):159–191, 2010.
- [12] P. Giordano. Order relation and geometrical representation of Fermat reals. submitted to *submitted to American Math. Journal*, 2010.
- [13] P. Giordano. The ring of Fermat reals. *Advances in Mathematics*, 225(4):2050–2075, 2010. DOI: 10.1016/j.aim.2010.04.010.
- [14] P. Iglesias-Zemmour. Diffeology. <http://math.huji.ac.il/~piz/documents/Diffeology.pdf>, July 9 2008.
- [15] A. Kock. *Synthetic Differential Geometry*, volume 51 of *London Math. Soc. Lect. Note Series*. Cambridge Univ. Press, 1981.
- [16] I. Kolár, P.W. Michor, and J. Slovák. *Natural operations in differential geometry*. Springer-Verlag, Berlin, Heidelberg, New York, 1993.

- [17] A. Kriegl and P.W. Michor. Product preserving functors of infinite dimensional manifolds. *Archivum Mathematicum (Brno)*, 32, 4:289–306, 1996.
- [18] A. Kriegl and P.W. Michor. *The Convenient Settings of Global Analysis*, volume 53 of *Mathematical Surveys and Monographs*. AMS, Providence, 1997.
- [19] R. Lavendhomme. *Basic Concepts of Synthetic Differential Geometry*. Kluwer Academic Publishers, Dordrecht, 1996.
- [20] T. Levi-Civita. Sugli infiniti ed infinitesimi attuali quali elementi analitici. *Atti del Regio Istituto Veneto di Scienze, Lettere ed Arti*, VII(4):1765–1815, 1893.
- [21] I. Moerdijk and G.E. Reyes. *Models for Smooth Infinitesimal Analysis*. Springer, Berlin, 1991.
- [22] M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations*, Pitman Research Notes in Mathematics Series **259**. Longman Scientific & Technical, Harlow 1992.
- [23] Z.M. Odibat and N.T. Shawagfeh. Generalized Taylor’s formula. *Applied Mathematics and Computation*, 186:286–293, 2007.
- [24] A. Robinson. *Non-standard analysis*. Princeton University Press, 1966.
- [25] K. Shamseddine. *New Elements of Analysis on the Levi-Civita Field*. PhD thesis, Michigan State University, East Lansing, Michigan, USA, 1999.
- [26] K. Shamseddine and M. Berz. Intermediate value theorem for analytic functions on a Levi-Civita field. *Bull. Belg. Math. Soc. Simon Stevin*, 14:1001–1015, 2007.
- [27] H. Vernaëve. Ideals in the ring of Colombeau generalized numbers. *Communications in Algebra*, 38(6):2199–2228, 2010.